Solving Seven Open Problems of Offline and Online Control in Borda Elections

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Abstract

Standard (offline) control scenarios in elections (such as adding, deleting, or partitioning either voters or candidates) have been studied for many voting systems, natural and less natural ones, and the related control problems have been classified in terms of their complexity. However, for one of the most important natural voting systems, the Borda Count, only a few such complexity results are known. We reduce the number of missing cases by pinpointing the complexity of three control scenarios for Borda elections, including some that arguably are among the practically most relevant ones. We also study online candidate control, an interesting dynamical, partial-information model due to Hemaspaandra et al. (2012a), who mainly focused on general complexity bounds by constructing artificial voting systems—only recently they succeeded in classifying four problems of online candidate control for one *natural* voting system: sequential plurality (Hemaspaandra et al. 2016). We settle the complexity of another four natural cases: constructive and destructive online control by deleting and adding candidates in sequential Borda elections.

Introduction

In computational social choice, voting has been studied comprehensively and profoundly, with applications ranging from political elections over recommender systems (e.g., to select movies according to certain criteria (Ghosh et al. 1999)) and webpage ranking algorithms (Dwork et al. 2001) to multiagent planning (Ephrati and Rosenschein 1993). Particular attention has been paid to using complexity as a barrier to tampering with election outcomes via manipulation, control, and bribery; recent book chapters by Conitzer and Walsh (2016), Faliszewski and Rothe (2016), and Baumeister and Rothe (2015) provide an overview.

We focus on standard (i.e., offline) and online control scenarios in elections. Control actions such as adding, deleting, or partitioning either voters or candidates have been studied for many voting systems—natural ones such as plurality, Condorcet, and approval voting (Bartholdi et al. 1992; Hemaspaandra et al. 2007), Copeland (Faliszewski et al. 2009), and Schulze voting (Parkes and Xia 2012) and less natural ones such as variants of approval and range voting (Erdélyi et al. 2009; Menton 2013)—and the related control problems have been classified in terms of their complexity. However, for one of the most prominent natural voting systems, the Borda Count, only a few such results are known.

Related Work. Bartholdi et al. (1989), Bartholdi and Orlin (1991), and Conitzer et al. (2007) were the first to study manipulation; see the chapter by Conitzer and Walsh (2016) for an overview. For *weighted* Borda elections, the complexity of coalitional manipulation was established early on by Conitzer et al. (2007). The *unweighted* case turned out to be more challenging; eventually, it was settled independently by Betzler et al. (2011) and Davies et al. (2011).

Control (and bribery) of elections is surveyed by Faliszewski and Rothe (2016) and Baumeister and Rothe (2015) who provide an extensive list of references. In particular, Bartholdi et al. (1992) introduced the *standard (or offline) constructive control settings* where, roughly, an election chair seeks to make her favorite candidate win by exerting structural changes to a given election, such as adding or deleting or partitioning either voters or candidates. Hemaspaandra et al. (2007) introduced the corresponding *destructive control settings* where the chair's goal is to prevent a given candidate's victory. Each of these scenarios has been thoroughly discussed in the literature (see, e.g., the above book chapters), along with real-world applications. However, except for plurality, not much is known about control for scoring protocols: Hemaspaandra et al. (2014) obtained a dichotomy result for constructive control by adding voters in scoring protocols; some recent results on veto are due to Maushagen and Rothe (2016); and the relatively few results obtained for Borda so far are due to Russel (2007), Elkind et al. (2011), Loreggia et al. (2015), and Chen et al. (2015)—see Table 1 for an overview of their results.

Another compelling line of research, focusing specifically on gerrymandering by analyzing geographic manipulation, has been introduced recently by Lewenberg and Lev (2016) (see also the related paper by Bachrach et al. (2016)).

Hemaspaandra et al. (2012a; 2012b; 2014; 2016) proposed interesting new models of online manipulation and online control in *sequential* elections. (A game-theoretic approach to sequential elections was considered earlier by Desmedt and Elkind (2010).) Unlike the standard manipulation model where all voters cast their votes simultaneously and the manipulators have complete knowledge of all vot-

control type	CAC		CDC		CPC-TE		CPC-TP		CRPC-TE		CRPC-TP		CAV		CDV		CPV-TE		CPV-TP	
	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D
Borda	R[$]	V[£]	R[❄]	V[£]	?	?	?	?	**R**[◇]	?	?	?	R[§]	V[§]	**R**[♠]	V[§]	**R**[♡]	V[§]	?	?

Table 1: Overview of offline-control complexity results for Borda. Our notation of control types is standard (Faliszewski and Rothe 2016). "R" means that Borda is resistant to this type of control, "V" means vulnerability to this type, and "?" means that this is an open question. Results in boldface are established in Thm. 1 (marked by ♠), Thm. 2 (♡), and Thm. 3 (◇). The other results are due to Russel (2007) (marked by §), Elkind et al. (2011) ($), Loreggia et al. (2015) (£), and Chen et al. (2015) (❄).

ers' preferences, in the *online manipulation* model (Hemaspaandra et al. 2014) voters cast their votes one after the other and manipulators can see the past votes but not the future ones, so the current manipulator must decide—*right in this moment when it is her turn to vote*—what vote to cast. The manipulators have an "ideal" ranking of candidates and their goal is, in the constructive variant, to ensure victory for some given candidate d or a better one in their ranking (or, in the destructive variant, to ensure that neither d nor any worse candidate wins), *no matter which votes the future voters will cast*. This "maximin" approach is inspired by the field of online algorithms (see, e.g., the book by Borodin and El-Yaniv (1998)). Similarly, in *online voter control* scenarios (Hemaspaandra et al. 2012b), voters cast their votes one after the other and now the chair must decide—*right in this moment when the current vote is cast*—whether or not to exert the control action at hand (e.g., either to delete the current vote now or not to delete it ever). Again, the chair's goal is either—in the constructive case—to ensure victory of some given candidate d or a better one in her ideal ranking, or—in the destructive case—to ensure that neither d nor any worse candidate wins. *Online candidate control* (Hemaspaandra et al. 2012a; 2016) is modeled similarly, except that now the candidates appear in order, one after the other, and the votes are gradually extended to include the current candidate when she steps forward. Hemaspaandra et al. (2012a; 2012b; 2014) provide PSPACE-completeness results for various specific scenarios in all three settings, albeit for artificially constructed voting systems. For specific scenarios of online manipulation and online voter control in sequential elections, they also consider *natural* voting systems, such as sequential plurality and veto, and provide a number of complexity results (each much lower than PSPACE-completeness). Recently, they also found polynomial-time algorithms solving four problems of online candidate control for the first *natural* voting system: sequential plurality (Hemaspaandra et al. 2016).

Our Contribution. Previously, only eight (offline) control scenarios have been studied for Borda elections: constructive/destructive control by deleting/adding candidates and by adding voters, and destructive control by deleting voters and by partition of voters in the so-called ties-eliminate (TE) model where winners of the two subelections resulting from a voter partition proceed to the final run-off only when they are unique (Hemaspaandra et al. 2007). For Borda, the above constructive control problems have been shown NP-complete, whereas the destructive control problems have been shown to be solvable in polynomial time (see Table 1).

We settle the complexity of offline control for three further control scenarios that have been left open in Borda elections (again, see Table 1). Note that two of these model real-world scenarios (namely, vote suppression and gerrymandering) that arguably are among the practically most relevant ones. Vote suppression describes various strategies to influence the outcome of an election by preventing voters to cast their votes, effectively deleting their votes from the election (and we may assume that the chair wants to delete as few votes as possible so as to avoid that the control action will be detected). This scenario is modeled by *constructive control by deleting voters*. *Constructive control by partition of voters* models gerrymandering—which refers to maliciously resizing voting districts—even though in a simplified variant: There is only one district that can be divided into two. However, if—as we will show for these constructive cases—complexity is high even in this simple setting, it will be at least as high for more involved models of gerrymandering (see, e.g., the models proposed by Erdélyi et al. (2015), Bachrach et al. (2016), and Lewenberg and Lev (2016)). Note further that, traditionally, control problems are defined for *unweighted* elections, which typically requires more involved constructions for proving hardness.

While there are several complexity results on online manipulation and online voter control in sequential elections for some natural voting systems (Hemaspaandra et al. 2012b; 2014), Hemaspaandra et al. (2012a) originally did not provide results on online *candidate* control in sequential elections for natural voting systems—only recently they succeeded in showing that sequential plurality elections are vulnerable to four types of online candidate control (Hemaspaandra et al. 2016), the first such results for a *natural* voting system. We establish another four such results for sequential Borda elections: for constructive and destructive online control by adding and by deleting candidates.

Preliminaries

An *election* (C,V) is given by a set C of candidates and a list V of votes, typically (and throughout this paper) assumed to be linear orders over the candidates. We will express a vote over C as a string giving the order of the candidates from the most preferred to the least preferred one; for example, if $C = \{a, b, c, d\}$, a vote $c\ b\ d\ a$ means that this voter prefers c to b, b to d, and d to a. A *voting system* is a rule that assigns a set of winners to each given election. Positional scoring rules are a particularly important class of voting rules, and among those we will consider only one most prominent member, the *Borda Count:* For m candidates, each candidate in position i of the voters' rankings scores $m-i$ points; the winners are those candidates scoring the most points. Let $score_{(C,V)}(x)$ be the number of points candidate x receives in election (C,V) according to Borda. Let $dist_{(C,V)}(x,y) = score_{(C,V)}(x) - score_{(C,V)}(y)$. For a subset $X \subseteq C$, we write $\overrightarrow{X}$ in a vote as a shorthand for the ranking of these candidates in an arbitrary but fixed order, and we write $\overleftarrow{X}$ as a shorthand for their ranking in reverse order.

In the next sections we will define the control types considered here; for other standard control types, see the book chapters by Faliszewski and Rothe (2016) and Baumeister and Rothe (2015) and the references therein. A voting system is *susceptible to a type of control* (e.g., constructive control by deleting voters) if there is an election for which the chair can reach her goal (e.g., turning a nonwinning candidate into a winner) by exerting this type of control. Borda Count is susceptible to each standard control type, including those considered here. If a voting system is susceptible to a type of control, one studies whether it is *vulnerable to it* (i.e., whether the associated control problem can be solved in polynomial time) or whether it is *resistant to it* (i.e., whether the associated control problem is NP-hard).

For offline control, we define our control problems in the *unique-winner model*, which means that a constructive (destructive) control action is considered successful only if the distinguished candidate is a *unique* winner (is *not a unique* winner). This model better fits the ties-eliminate (TE) rule for control-by-partition scenarios than the ties-promote (TP) rule, according to which *all* subelection winners proceed to the final run-off (Hemaspaandra et al. 2007). Using the same constructions and slightly modified arguments, our proofs also work in the *nonunique-winner model*. In this model, for a constructive (destructive) control action to be successful, the distinguished candidate is required to be only *a* winner (to be *not even a* winner). For online control, we adopt the nonunique-winner model, just as Hemaspaandra et al. (2012a; 2012b; 2014; 2016) do.

Offline Control in Borda Elections

In this section, we solve three open problems for (offline) control in Borda elections, starting with the following:

Borda-CCDV. Given a voting system $\mathscr{E}$, the problem $\mathscr{E}$-CONSTRUCTIVE-CONTROL-BY-DELETING-VOTERS ($\mathscr{E}$-CCDV) is defined as follows (Bartholdi et al. 1992): Given an election (C,V), a distinguished candidate p in C, and a nonnegative integer k, can p be made the unique winner of an election resulting from (C,V) by deleting at most k votes? In the proof of Theorem 1 we reduce from EXACT-COVER-BY-3-SETS (X3C) that is well known to be NP-complete (Garey and Johnson 1979): Given a set $X = \{x_1, \dots, x_{3k}\}$ and a family of subsets of X, $\mathscr{S} = \{S_1, \dots, S_n\}$, each with three elements, does there exist an exact cover of X, i.e., a subfamily $\mathscr{S}' \subseteq \mathscr{S}$ with $|\mathscr{S}'| = k$ such that each element $x_i \in X$ occurs in exactly one subset $S_j \in \mathscr{S}'$?

Theorem 1 *Borda is resistant to constructive control by deleting voters.*

Proof. Membership of Borda-CCDV in NP is obvious. To prove NP-hardness, we now describe a reduction from X3C to Borda-CCDV. Let $(X, \mathscr{S})$ be a given X3C instance with $X = \{x_1, \dots, x_m\}$, $m = 3k$, $k > 1$, and $\mathscr{S} = \{S_1, \dots, S_n\}$ with $S_i \subseteq X$ and $|S_i| = 3$ for each i, $1 \leq i \leq n$. Construct from $(X, \mathscr{S})$ a Borda-CCDV instance $((C,V), p, k)$ as follows. The candidate set is $C = X \cup B \cup \{p\}$ with $B = \{b_1, \dots, b_{3m^2}\}$ and distinguished candidate p the chair wants to make a unique winner. The list V of votes is constructed as follows:

1. For each i, $1 \leq i \leq m$, there are m votes $x_i\ \overrightarrow{X \setminus \{x_i\}}\ p\ \overrightarrow{B}$ and m votes $x_i\ p\ \overleftarrow{X \setminus \{x_i\}}\ \overleftarrow{B}$.
2. For each j, $1 \leq j \leq n$, there is one vote $v_j = \overrightarrow{S_j}\ \overrightarrow{B}\ \overrightarrow{X \setminus S_j}\ p$ and one vote $w_j = p\ \overleftarrow{X \setminus S_j}\ \overleftarrow{B}\ \overleftarrow{S_j}$.

p is not a Borda winner of election (C,V), since $dist_{(C,V)}(p, x_i) = -m(m+1) = -(m^2+m)$ for each i, $1 \leq i \leq m$. Note that for each candidate $b_j \in B$ we have $dist_{(C,V)}(p, b_j) = m \cdot m(|B| + (m-1) + 1) = 3m^4 + m^3$. To win the election, p needs to make up a deficit of $m^2 + m$ points for each $x_i \in X$. We claim that $(X, \mathscr{S})$ is in X3C if and only if $((C,V), p, k)$ is in Borda-CCDV.

From left to right, suppose there is an exact cover $\mathscr{S}' \subseteq \mathscr{S}$. Let $V' = \{v_j \mid S_j \in \mathscr{S}'\}$, $|V'| = k$. Consider the election $(C, V \setminus V')$. By deleting the votes in V' from (C,V), p doesn't lose any points because p is ranked last in these votes. However, every $x_i \in X$ loses at least $|B| + m - 2 = 3m^2 + m - 3$ points once (for deleting the v_j with $x_i \in S_j$) and at least one point in each of the $k-1$ other deleted votes. Hence, we have $dist_{(C,V\setminus V')}(p, x_i) \geq -(m^2+m) + 3m^2 + m - 3 + k - 1 = 2m^2 + k - 4 > 0$ for each $x_i \in X$. Since also the candidates from B are losing points by this deletion, p still is better off than each $b_j \in B$, i.e., we have $dist_{(C,V\setminus V')}(p, b_j) > 0$, which makes p the unique winner of $(C, V \setminus V')$.

From right to left, suppose that p can be made a unique winner by deleting at most k votes from (C,V). Deleting k votes of the form $x_i\ \overrightarrow{X \setminus \{x_i\}}\ p\ \overrightarrow{B}$ or $x_i\ p\ \overleftarrow{X \setminus \{x_i\}}\ \overleftarrow{B}$ implies $dist(p, x_i) \leq -(m^2+m) + k \cdot m = -(m^2+m) + \frac{m^2}{3} < 0$, so p would still score fewer points than the candidates in X. And deleting a vote w_j for some j, $1 \leq j \leq n$, would harm p (who is ranked on top of these votes) even more than the x_i. However, by deleting a vote v_j for some j, $1 \leq j \leq n$, p reduces her deficit against the three candidates in S_j by at least $3m^2 + m - 3$ points and against the other candidates in X by at most $m-2$ points. Hence, exactly k votes must be deleted to make p a unique Borda winner; let $V' = \{v_{a_1}, \dots, v_{a_k}\}$ be this set. Since we have $dist_{(C,V\setminus V')}(p, x_f) \leq -(m^2+m) + k(m-2) = -(m^2+m) + \frac{m^2-2m}{3} < 0$ for each $x_f \in X$ with $x_f \notin \bigcup_{1 \leq i \leq k} S_{a_i}$, but $dist_{(C,V\setminus V')}(p, x_g) \geq -(m^2+m) + 3m^2 + m - 3 = 2m^2 - 3 > 0$ for each $x_g \in X$ with $x_g \in \bigcup_{1 \leq i \leq k} S_{a_i}$, p would defeat all these x_g but none of those x_f. That is, p

can defeat $x_i \in X$ only by deleting some voter v_j, $1 \leq j \leq n$, such that $x_i \in S_j$. Since p must defeat all such candidates by deleting k such votes, there exists an exact cover. ❑

Borda-CCPV-TE. For a voting system $\mathscr{E}$, we now turn to $\mathscr{E}$-CONSTRUCTIVE-CONTROL-BY-PARTITION-OF-VOTERS-TE ($\mathscr{E}$-CCPV-TE), as defined by Bartholdi et al. (1992), in the ties-eliminate model due to Hemaspaandra et al. (2007): Given an election (C,V) and a candidate p in C, can V be partitioned into V_1 and V_2 such that p is the unique $\mathscr{E}$ winner of the two-stage election where only unique $\mathscr{E}$ winners of subelections (C,V_1) and (C,V_2) proceed to the final run-off? We will make use of a reduction from the well-known NP-complete problem PARTITION (Garey and Johnson 1979): Given a set $A = \{1,\dots,n\}$ and a list $s = (s_1,\dots,s_n)$ of nonnegative integers, can A be partitioned into two subsets A_1 and A_2 such that $\sum_{i\in A_1} s_i = \sum_{i\in A_2} s_i$?

Theorem 2 *Borda is resistant to constructive control by partition of voters in the ties-eliminate model.*

Proof. Obviously, Borda-CCPV-TE is in NP. To prove NP-hardness, we now provide a reduction from PARTITION to Borda-CCPV-TE. Given a PARTITION instance (A,s) with $A = \{1,\dots,n\}$, a list $s = (s_1,\dots,s_n)$ of nonnegative integers, and $K = \sum_{i\in A} s_i$, construct a Borda-CCPV-TE instance $((C,V),p)$ as follows. Let $C = B^{(1)} \cup \dots \cup B^{(n)} \cup D \cup T \cup \{p,r,r^*\}$ (with p being the distinguished candidate the chair wants to make a unique winner) contain n sets $B^{(i)} = \{b_1^{(i)},\dots,b_{2s_i-1}^{(i)}\}$, $1 \leq i \leq n$, a set $D = \{d_1,\dots,d_{2K}\}$, and a set $T = \{t_1,\dots,t_{2K+2}\}$.

As a notation, we write $D_{i,j} = \{d_i,d_{i+1},\dots,d_{j-1},d_j\}$, where $1 \leq i \leq j \leq 2K$. Construct V to consist of $n+2$ votes:

$$v_i = r\,B^{(i)}\,r^*\,p\,T\,D\,B^{(1)}\cdots B^{(i-1)}\,B^{(i+1)}\cdots B^{(n)},\ i \in A,$$
$$v_{n+1} = p\,D_{1,K}\,r^*\,D_{K+1,2K}\,r\,T\,B^{(1)}\cdots B^{(n)},$$
$$v_{n+2} = r^*\,D_{1,K-1}\,r\,T\,p\,D_{K,2K}\,B^{(1)}\cdots B^{(n)}.$$

Since $dist_{(C,V)}(p,r) = 2K+2-(2K+3)-(2K+2n) < 0$, p is not a Borda winner of (C,V). We claim that (A,s) is in PARTITION if and only if $((C,V),p)$ is in Borda-CCPV-TE.

From left to right, suppose there is a partition of A into two sets, A_1 and A_2, such that $\sum_{i\in A_j} s_i = K/2$ for $j \in \{1,2\}$. Assign v_{n+1} to V_1 and v_{n+2} to V_2. Add v_i to V_1 for each $i \in A_1$, and add the remaining votes v_j with $j \in A_2$ to V_2.

In subelection (C,V_1), p scores $K+1$ points more than r^* and $2K+2$ points more than r due to vote v_{n+1} alone. Candidate r^* scores at most $n-1$ points more than p by the other votes in V_1, so $dist_{(C,V_1)}(p,r^*) \geq K+1-(n-1) > 0$, since $K \geq n$ (if $s_1 = \dots = s_n = 1$, we have $K = n$, otherwise we have $K > n$.) And r scores $2s_i$ points more than p for each vote $v_i \in V_1$ with $1 \leq i \leq n$. Since $\sum_{i\in A_1} s_i = K/2$, $dist_{(C,V_1)}(p,r) = 2K+2-2K/2 = K+2 > 0$, so p scores more points than r and r^* in (C,V_1). Note also that p is preferred to all candidates of D and T in the votes of V_1. A candidate $b_j^{(i)}$ is preferred to p in at most one vote in V_1 and thus can score at most $2\max_{i\in A}\{s_i\}$ points more than p. However, p scores at least $|T| = 2K+2$ points more than $b_j^{(i)}$ in v_{n+1} and thus has a higher score in total than $b_j^{(i)}$ in (C,V_1) because $K \geq \max_{i\in A}\{s_i\}$. It follows that p is the unique Borda winner of subelection (C,V_1) and proceeds to the final run-off.

In subelection (C,V_2), r^* scores K points more than r due to vote v_{n+2} alone. By the other votes in V_2, however, r scores $2K/2 = K$ points more than r^*, since $\sum_{i\in A_2} s_i = K/2$. Thus $dist_{(C,V_2)}(r,r^*) = K-K = 0$, so r and r^* are tied in (C,V_2). In the votes from V_2, (a) both r and r^* are preferred to p and to all candidates from T, (b) r is preferred to each $b_j^{(i)} \in B^{(i)}$, and (c) r^* is preferred to each $d_j \in D$. Overall, both r and r^* win subelection (C,V_2) and thus are both eliminated by the tie-handling rule. It follows that no candidate moves forward to the final run-off from subelection (C,V_2).

Being the only participant, p alone wins the run-off.

From right to left, suppose now that p can be made the only Borda winner by some partition of V into V_1 and V_2. Thus p is the only Borda winner of at least one of the subelections (C,V_1) and (C,V_2). Without the vote v_{n+1}, however, p cannot win a subelection, since both r and r^* are preferred to p in all other votes. Let (C,V_1) be the subelection (with $v_{n+1} \in V_1$) that p is the only Borda winner of. Note that $v_{n+2} \notin V_1$, since otherwise p would lose too many points compared to r and r^* that cannot be regained via votes v_i, $1 \leq i \leq n$. Thus $v_{n+2} \in V_2$. Due to the tie-handling rule, at most two candidates can take part in the final run-off. In direct comparison, p is defeated by r and r^*, since $dist_{(\{p,r\},V)}(p,r) = -n$ and $dist_{(\{p,r^*\},V)}(p,r^*) = -n$. Therefore, also some votes v_i, $1 \leq i \leq n$, must belong to V_2, for otherwise r^* would win (C,V_2) and would then defeat p in the run-off. Since neither candidates from D nor T nor some $b_j^{(i)} \in B^{(i)}$ can win (C,V_2) by adding votes v_i, $1 \leq i \leq n$, to V_2, r and r^* must tie so as to make sure that no candidate can proceed from (C,V_2) to the final run-off. We have $dist_{(C,\{v_{n+2}\})}(r,r^*) = -K$ and $dist_{(C,\{v_i\})}(r,r^*) = 2s_i$ for each i, $1 \leq i \leq n$. Thus we need to have $dist_{(C,V_2)}(r,r^*) = -K + \sum_{v_i\in V_2} 2s_i$. Hence, $dist_{(C,V_2)}(r,r^*) = 0$ requires $\sum_{v_i\in V_2} 2s_i = 2\sum_{v_i\in V_2} s_i = 2K/2 = K$ to hold. Let $A_2 = \{i \mid v_i \in V_2\}$, so $\sum_{i\in A_2} s_i = \sum_{v_i\in V_2} s_i = K/2$, and with $A_1 = A \setminus A_2$ we obtain a partition of A such that $\sum_{i\in A_1} s_i = \sum_{i\in A_2} s_i = K/2$. ❑

Borda-CCRPC-TE. For a voting system $\mathscr{E}$, we consider the problem $\mathscr{E}$-CONSTRUCTIVE-CONTROL-BY-RUN-OFF-PARTITION-OF-CANDIDATES-TE ($\mathscr{E}$-CCRPC-TE) in which we ask, given an election (C,V) and distinguished candidate $p \in C$, whether the candidate set C can be partitioned into two subsets C_1 and C_2 such that p is the unique Borda winner of the final run-off among the Borda winners of subelections (C_1,V) and (C_2,V) (again, only unique subelection winners move forward). We will make use of a reduction from 3SAT, the standard NP-complete satisfiability problem (Garey and Johnson 1979): Given a boolean formula φ in 3-CNF (i.e., with exactly three literals per clause), does there exist a satisfying truth assignment to φ? For a boolean formula φ, we denote by $\#_i$ the number of literals occurring in the ith clause that are negated variables.

Theorem 3 *Borda is resistant to constructive control by run-off partition of candidates in the ties-eliminate model.*

Proof. Again, membership of Borda-CCRPC-TE in NP is obvious. To show NP-hardness, we now provide a reduction from 3SAT to Borda-CCRPC-TE. Given a 3SAT instance $\varphi(x_1, x_2, \ldots, x_n)$, construct a Borda-CCRPC-TE instance $((C,V),p)$ as follows. Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of variables and let $K = \{K_1, K_2, \ldots, K_m\}$ be the set of clauses of φ, where $K_i = (\ell_i^{(1)} \vee \ell_i^{(2)} \vee \ell_i^{(3)})$, $1 \le i \le m$. Furthermore, let $D = \{d_1, d_2, d_3\}$ and $D_i = \{d_j \mid 1 \le j \le i\} \subseteq D$. Define the candidate set by $C = X \cup K \cup \{p, r, r^*\} \cup D$ with p being the distinguished candidate the chair wants to make a unique winner. Define V to consist of the following votes:

1. For each i, $1 \le i \le m$, there are two votes: $\overrightarrow{C \setminus \{p, K_i\} \cup D_{\#_i}}\ p\ D_{\#_i}\ K_i$ and $K_i\ p\ \overleftarrow{C \setminus \{p, K_i\} \cup D_{\#_i}}\ D_{\#_i}$.
2. For each i, $1 \le i \le m$, and for each literal $\ell_i^{(1)}$, $\ell_i^{(2)}$, and $\ell_i^{(3)}$, there are two votes: either $K_i\ x_j\ p\ \overrightarrow{C \setminus \{K_i, x_j, p\}}$ and $\overleftarrow{C \setminus \{K_i, x_j, p\}}\ p\ K_i\ x_j$ if $\ell_i^k = \overline{x_j}$ is a negated variable, or $\overrightarrow{C \setminus \{K_i, x_j, p\}}\ p\ x_j\ K_i$ and $x_j\ K_i\ p\ \overleftarrow{C \setminus \{K_i, x_j, p\}}$ if $\ell_i^k = x_j$ is a positive variable.
3. There are m votes of the form $r^*\ r\ \overrightarrow{K}\ \overrightarrow{D}\ p\ X$ and m votes of the form $r\ p\ \overleftarrow{D}\ \overleftarrow{K}\ r^*\ X$.

Since $dist_{(C,V)}(p,r) = m(-5-m-1) = -m(m+6) < 0$, p does not win in (C,V). Note that p and r score the same number of points in the first two groups of votes. Later on, we will also need the following argument. Consider a clause candidate K_i. In the first group of votes, p scores as many points more than K_i as there are negated variables in clause K_i, namely $\#_i$. In the second group of votes, p gains one more point with respect to candidate K_i for each positive variable in clause K_i, and p loses one point with respect to candidate K_i for each negated variable in clause K_i. Since p and K_i score the same number of points in the third group of votes, we have $dist_{(C,V)}(p,K_i) = \#_i - \#_i + (3 - \#_i) = 3 - \#_i$. Assuming that one variable candidate x_j is assigned to the other subelection than p and K_i, if x_j is a negated variable in clause K_i then p gains one point with respect to candidate K_i, and if x_j is a positive variable in clause K_i then p loses one point with respect to K_i. Further, if C' is the set of candidates obtained by removing from C all variable candidates corresponding to positive variables in clause K_i, then $dist_{(C',V)}(p,K_i) = 3 - \#_i - (3 - \#_i) = 0$ because p is losing as many points with respect to K_i as there are positive variables in clause K_i. That is, p and K_i are tied in their subelection if (a) all variable candidates corresponding to positive variables in clause K_i are removed from the subelection containing p and K_i (and are assigned to the other subelection) and (b) all variable candidates corresponding to negated variables in clause K_i remain in the subelection with p and K_i. Therefore, for p to defeat K_i, either the subelection containing them also contains at least one variable candidate corresponding to a positive variable in clause K_i, or the other subelection contains at least one variable candidate corresponding to a negated variable in clause K_i, or both. We claim that φ is a yes-instance of 3SAT if and only if $((C,V),p)$ is a yes-instance of Borda-CCRPC-TE.

From left to right, suppose there is a satisfying truth assignment α to the variables of $\varphi(x_1, \ldots, x_n)$. Partition C into C_1 and C_2 so that C_2 contains r, r^*, and all variable candidates that are set to *false* in α, and C_1 contains all the other candidates. r and r^* tie in subelection (C_2,V) and are eliminated by the tie-handling rule. In (C_1,V) (C_1 consisting of p, D, K and all variable candidates that are set to *true* in α) p beats all other candidates (in particular, the clause candidates) because α is a satisfying truth assignment. Therefore, p wins the final election as she is the only candidate left.

From right to left, suppose p is the unique overall Borda winner for some partition of the candidates. r had to be eliminated in the first-round subelection; otherwise, r would have beaten p in the run-off. This can only be achieved by r^*, who can tie (but not beat) r in this subelection only if neither p nor the candidates in D or K are participating. Thus C_2 contains r, r^*, and some variable candidates, and C_1 contains p, all candidates from D and K, and the remaining variable candidates. In subelection (C_2,V), all winners are tieing and, therefore, are eliminated by the tie-handling rule. Since p is the only winner of her subelection, (C_1,V), and the run-off, the variable candidates must have been distributed among C_1 and C_2 according to the argument mentioned above. This leads to a satisfying truth assignment if every variable candidate in C_1 is assigned to *true*, and all the others to *false*. ❑

Online Control in Sequential Borda Elections

Finally, we turn to online candidate control in sequential Borda elections. We first describe the model and the related problems that are due to Hemaspaandra et al. (2012a; 2016), who also provide motivating examples for these control scenarios in detail, ranging from TV singing/dancing talent shows to university faculty-hiring processes. Specifically, we restrict ourselves to formalizing *online constructive control by deleting candidates for Borda*, denoted by online-Borda-CCDC. The corresponding problem for adding candidates (online-Borda-CCAC) and their destructive counterparts (online-Borda-DCDC and online-Borda-DCAC) can be defined analogously. Capturing the election chair's "moment of decision," an input to online-Borda-CCDC encodes the history of the sequential election process up to a given point in time and specifically consists of: the candidate set C, the set of voters V, the chair's ideal ranking σ of the candidates, a distinguished candidate $d \in C$, an order in which the candidates will show up, with flags indicating who the current candidate is and which of the previous candidates have been deleted, the voters' preferences masked down to the still-standing (i.e., already revealed but not deleted) candidates, and a nonnegative integer bound k on how many deletions are left for the chair to use. The question the chair now faces is whether she has a "forced win" by either deleting or not deleting the current candidate right in this moment (she will never again have this choice about this current candidate), where by "forced win" we mean whether the set $\{c \mid c \ge_\sigma d\}$ will contain a Borda winner eventually, *no matter what voter preferences will be revealed about the future candidates who have not shown up yet*. For *destructive* online control by deleting candidates, Hemaspaandra et al. (2012a; 2016) distinguish the *non-hand-tied chair model* where the chair may delete some but not all candidates "*d* or worse" and the *hand-tied chair model* where the chair may never delete any candidate "*d* or worse."

Theorem 4 *online-Borda-*CCDC*, online-Borda-*DCDC *(both in the non-hand-tied and the hand-tied chair model), online-Borda-*CCAC*, and online-Borda-*DCAC *are in* P.

Proof. We restrict ourselves to sketching a proof that online-Borda-CCDC is in P. The other proofs are similar.

Given an input to online-Borda-CCDC as described above (using the same notation, e.g., d denoting the distinguished candidate and σ the chair's ideal ranking of the candidates), we give a polynomial-time algorithm that decides whether the chair has a forced win by either deleting or not deleting the current candidate, c. In fact, since the chair is facing these two options (to delete or not to delete c—unless the number of allowed deletions is used up already in which case c must be left in) now, our algorithm (to be described below) will be run twice, first pretending the chair's decision were to leave c in, then pretending the chair's decision were to remove c, and if at least one run yields a forced win for the chair, the input is accepted; otherwise it is rejected.

We call each $e \in C$ with $e \geq_\sigma d$ a *good* candidate and each $e \in C$ with $e <_\sigma d$ a *bad* candidate. Let b be the number of future (i.e., as yet unrevealed) bad candidates and let g be the number of future good candidates. Recall that all votes at this point are masked down to the still-standing candidates (but will be gradually extended when new candidates show up). Our polynomial-time algorithm now works as follows. If there is no voter, every candidate still standing in the end is a Borda winner with score zero, so we accept if there is a good candidate among those, and otherwise we reject. Further, in case all candidates have been revealed in the current moment, we simply determine their scores, and we accept if a good candidate has the highest score; otherwise, we reject. So from now on we may assume that there is at least one voter and not all candidates have shown up yet. We now determine the scores of all already revealed but not deleted candidates. If no good candidate has currently the highest score, the chair does not have a forced win: It may happen, for instance, that all future candidates will be ranked below all currently revealed candidates in the completed votes in the end, which would mean that all currently revealed candidates score the same number of points more than now and they each score more points than any future candidate, since these are ranked lower in each vote, so it is still true that no good candidate is a Borda winner, and we reject. (In this case, it doesn't matter whether future candidates will be deleted or not.)

Consider now the case that at least one good candidate is currently winning. Let $k \geq 0$ be the number of deletions left for the chair to use. If $k < b$ then there is at least one future bad candidate that cannot be removed. In the worst case, one such candidate ends up in the top positions of all completed votes in the end and thus is the only Borda winner, so the chair does not have a forced win and we reject. If $k \geq b$, however, all future bad candidates can be deleted by the chair, who then is left with $k - b \geq 0$ remaining possible deletions. If none of the previously revealed candidates is bad, only good candidates remain in the election and at least one of them will be a Borda winner in the end, so the chair has a forced win and we accept. Therefore, we now consider the final case that a good candidate currently has the highest score, all future bad candidates can be deleted with $k-b \geq 0$ possible deletions remaining for the chair, and at least one bad candidate was previously revealed and not deleted (and so will remain in the election). For each future good candidate who will not be deleted, every other candidate can score at most one additional point in each vote, depending on their relative position in the votes. In order to spoil a forced win for the chair, some previously revealed and not deleted bad candidate would have to score enough additional points due to the $g-(k-b)$ future good candidates that cannot be deleted so as to have more points than each good candidate in the end.

Let i be a bad candidate still in the election and let s be a future good candidate. When s is being revealed, then i makes up one point with respect to a good candidate j if there is a vote of the form $\cdots\ i\ \cdots\ s\ \cdots\ j\ \cdots$. That is, if such a bad candidate's deficit regarding the good candidates is not too large and there are sufficiently many votes of this form for all good candidates, the bad candidate can still become a unique Borda winner in the worst case (for the chair). We may assume that the revealed good candidates that won't be deleted, call them $s_1, \ldots, s_{g-(k-b)}$, occur directly behind the bad candidate i in these votes: $\cdots\ i\ s_1\ \cdots\ s_{g-(k-b)}\ \cdots$. For a good candidate j, let $v_{i,j}$ be the number of votes in which i precedes j. Then i can make up $v_{i,j} \cdot (g-(k-b))$ points with respect to j in the worst case. From the remaining $|V| - v_{i,j}$ votes, both i and j would gain the same number of points. Therefore, all we need to check is whether there is a bad candidate i still in the race such that for all good candidates j currently in the election, $score(i) + v_{i,j} \cdot (g-(k-b)) > score(j)$, where $score(h)$ denotes candidate h's current score. If so, i can become a unique Borda winner in the end, which spoils the chair's forced win, so we reject. Otherwise, for each bad candidate there is a good candidate whose score is at least as high in the end, even if the candidates still to be revealed and not deleted will be in the worst positions for this good candidate: The chair has a forced win and we accept. ❑

Conclusions and Future Work

We have solved three open problems about the complexity of control in Borda elections, including two that may be considered the practically most relevant ones as they model vote suppression and gerrymandering. There are still some questions open for control in Borda, namely for partition of voters and for partition and run-off partition of candidates, each in the TP model. More generally, settling the complexity of control for all scoring rules, ideally obtaining dichotomy results, is a challenging task. We have also solved four cases of online candidate control in sequential Borda elections. We propose to establish more such results for Borda and other *natural* voting systems in this compelling model.

Acknowledgments. We thank the reviewers and the SPC member and acknowledge support by DFG RO 1202/15-1.

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