



Control complexity in Bucklin and fallback voting: A theoretical analysis [☆]



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ABSTRACT

Electoral control models ways of changing the outcome of an election via such actions as adding, deleting, or partitioning either candidates or voters. To protect elections from such control attempts, computational complexity has been used to establish so-called *resistance* results. We show that fallback voting, an election system proposed by Brams and Sanver [12] to combine Bucklin with approval voting, displays the broadest control resistance currently known to hold among natural election systems with a polynomial-time winner problem. We also study the control complexity of Bucklin voting and show that it performs almost as well as fallback voting in terms of control resistance. Furthermore, we investigate the parameterized control complexity of Bucklin and fallback voting, according to several parameters that are often likely to be small for typical instances. In a companion paper [28], we challenge our worst-case complexity results from an experimental point of view.

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1. Introduction

Since the seminal paper of Bartholdi et al. [5], the computational complexity of *electoral control* has been studied for a variety of voting systems. Unlike *manipulation* [4,3,16], which models attempts of strategic voters to influence the outcome of an election by casting insincere votes, *control* models attempts of an external actor, the “chair,” to tamper with an election’s participation structure so as to alter its outcome via such actions as adding, deleting, or partitioning either candidates or voters. A third way of tampering with the outcome of elections is *bribery* [32,34], which shares with manipulation the feature that votes are being changed, and with control the aspect that an external actor tries to change the outcome of an election. (We do not here investigate resistance to bribery.) Faliszewski et al. [35,33] and Conitzer [14] comprehensively survey known complexity results for control, manipulation, and bribery for various voting systems; Faliszewski et al. [29] do so with a focus on manipulation; and Baumeister et al. [6] do so with a particular emphasis on approval voting and its variants.

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Elections have been used for preference aggregation not only in the context of politics and human societies, but also in artificial intelligence, especially in multiagent systems, and other applied settings in computer science; see, e.g., [22, 42, 19]. In general, information increasingly arises from multiple perspectives, and must be collated—one way of thinking about election systems that points to the rich range of applications. The investigation of the computational properties of voting systems is thus well-motivated and the robustness and resistance to manipulation, control, and bribery in voting is of fundamental interest, but how can this be investigated and such desirable properties be established, evidenced, or indicated?

In their path-breaking papers, Bartholdi et al. [4, 5] proposed to employ computational complexity to explore this issue: If the problem of deciding whether an election can be tampered with in the manipulation or control scenario at hand is NP-hard, then this is evidence that the system is intrinsically difficult to manipulate or to control in this scenario, because of the computational intractability of mounting an attack. In this perspective, a central quest that has emerged in the past two decades of research is to find natural voting systems with polynomial-time winner determination that are computationally resistant to as many of the common control types as possible, where *resistance* means that the corresponding control problem is NP-hard. Each control type is either *constructive* (the chair of the election seeks to insure that some favored candidate wins) or *destructive* (the chair seeks to insure that some despised candidate does not win).

We study the control complexity of fallback voting, an election system introduced by Brams and Sanver [12] as a way of combining Bucklin and approval voting. We prove that fallback voting is resistant to each of the common types of control except two (namely, it is not resistant to destructive control by either adding or deleting voters), and we show that it is vulnerable (i.e., the corresponding control problem is in P) to these two control types. With these control resistances, fallback voting displays the broadest control resistance currently known to hold among natural election systems with a polynomial-time winner problem. In particular, fallback voting is fully resistant to constructive control and it is fully resistant to candidate control.

As the two control types fallback voting is vulnerable to are *destructive* types and as destructive control intuitively can be seen as less important than constructive control, one may now view the original line of research that was started by Bartholdi et al. [5] two decades ago as “satisfyingly answered,” and this is one of the main contributions of this paper. That is not to say that, among natural voting systems with polynomial-time winner determination, fallback voting were the one and only system with the strongest or broadest control resistance. Indeed, shortly after our results on fallback voting were made public in a technical report [27] dated March 11, 2011, Menton [55] reported analogous results for normalized range voting.¹ It is very well possible that also other voting systems have the same resistances and vulnerabilities as fallback voting, and there might even be such a system that in addition is resistant to the two types of destructive control where fallback voting lacks resistance. We merely mean that fallback voting is the first natural voting system with polynomial-time winner determination shown to display such an almost complete control resistance.

Fallback voting is a hybrid system combining approval and Bucklin voting, and it is clear that each of these two constituent “pure” systems are certainly more natural than their hybrid. Therefore, we also study the control complexity of Bucklin voting itself. While many important voting systems—including plurality, Condorcet, Copeland, maximin, and approval voting—have already been investigated with respect to electoral control (see the references in “Related Work” below), Bucklin voting is one of the few central voting systems for which a thorough study of the control complexity has been missing to date. We show that Bucklin voting has no more than one control resistance fewer than fallback voting (namely, possibly, regarding destructive control to partition of voters in the tie-handling model TP, see Section 2.2 for the definition). In particular, Bucklin voting is also fully resistant to constructive control and fully resistant to candidate control. Since Bucklin voting is a special case of fallback voting, each resistance result for Bucklin voting strengthens the corresponding resistance result for fallback voting.

Now that the line of research investigating electoral control resistance issues in terms of NP-hardness is “satisfyingly answered” with the results in this paper, it is appropriate to revisit the roots of this research line. In particular, what do resistance results based on NP-hardness results actually mean in practice, that is, for “typical” elections? We explore this question in two ways, one theoretical and one experimental—both meriting further investigation in understanding the robustness and resistance of various voting systems to malicious meddling. Our experimental analysis can be found in the companion paper [28] that is based on [62].

On the theory side, we study the parameterized complexity of some of the key control problems in natural parameterizations. Parameterized complexity allows a more fine-grained deployment of computational complexity, having the ability to model more closely “typical input structure.”² In particular, we obtain W[2]-hardness results for problems related to adding/deleting either candidates or voters, parameterized by the number of candidates/voters that have been added/deleted.

Related work The study of electoral control was initiated by Bartholdi et al. [5], who introduced a number of constructive control types and investigated plurality and Condorcet voting in this regard. The common types of destructive control

¹ Historical aside: The version of his technical report [54] that establishes a matching number of resistances is dated April 25, 2011.

² We stress that “typical input structure” is different from (and should not be confused with) the “typical elections” used in our experimental results [28, 62]. While the former refers to typical parameters in the parameterized control problems we study, the latter refers to generating typical elections according to some distribution model.

were proposed by Hemaspaandra et al. [43], who studied destructive control for plurality and Condorcet voting and constructive and destructive control for approval voting. Plurality voting was the first natural system (among those having a polynomial-time winner problem) found to be fully resistant to candidate control. Faliszewski et al. [34] studied the control complexity of the whole family of Llull/Copeland voting systems and were the first to find a natural voting system that is resistant to all common types of constructive control.³ Another such voting system with full resistance to constructive control, and also to candidate control, is sincere-strategy preference-based approval voting (SP-AV), as shown by Erdélyi et al. [25]. Prior to this paper, SP-AV was the voting system displaying the broadest control resistance among natural systems with a polynomial-time winner problem. However, SP-AV (as modified by Erdélyi et al. [25]) is arguably less natural a system than fallback voting.⁴ Note also that plurality has fewer resistances to voter control and Copeland voting has fewer resistances to destructive control than either of Bucklin and fallback voting. If we disregard SP-AV for the reasons mentioned in Footnote 4, all “natural” systems with polynomial-time winner determination (whose control behavior has been studied previously) are vulnerable to considerably more control types than Bucklin or fallback voting: plurality to six types, both Copeland and Condorcet to seven types, Llull to eight types, and approval voting to nine types. As mentioned above, shortly after our results were made public, Menton [55] showed that normalized range voting has the same number of resistances as fallback voting. See also the bookchapter of Faliszewski and Rothe [30] for a survey on these results.

The parameterized complexity of electoral control has been studied by Betzler and Uhlmann [8] and Faliszewski et al. [34] for Llull/Copeland voting, by Liu et al. [52] for plurality, Condorcet, and approval voting, and by Liu and Zhu [51] for maximin voting.

Faliszewski et al. [37] study control for a more flexible type of attack (so-called “multimode control”) and focus more on “combined” vulnerability than resistance. Meir et al. [53] consider a different type of control scenario as well, using utility functions rather than constructive/destructive control, and they restrict their attention to adding/deleting candidates/voters. Both papers just mentioned study different voting systems. Recently, Hemaspaandra et al. [45,46] investigated *online* control in *sequential* elections, which is quite a different model than standard control in simultaneous elections studied here, see also their related work [48] on online manipulation in sequential elections.

As mentioned earlier, manipulation is related to, but different from control and has been studied even more extensively, in particular by Bartholdi et al. [4], Bartholdi and Orlin [3], Conitzer et al. [16], Conitzer and Sandholm [15], Elkind and Lipmaa [21], Bachrach et al. [1], Obratzsova et al. [58], Faliszewski et al. [36,31], Isaksson et al. [50], Mossel and Racz [56], Xia et al. [66], Xia and Conitzer [64,65], Faliszewski et al. [38], Brandt et al. [13], and Elkind and Erdélyi [20], who follow up the approach to consider voting rule uncertainty proposed by Baumeister et al. [7] for the more general possible winner problem. Much of this work has been surveyed by Faliszewski et al. [35,29], Conitzer [14], Baumeister et al. [6], and Faliszewski et al. [33]. Among the recent highlights regarding manipulation are the papers by Betzler et al. [9] and Davies et al. [17], who built on earlier work by Xia et al. [67] to show that Borda voting is NP-hard to manipulate, even with only two manipulators and even in the unweighted case.

Note that all our NP-hardness results regard control problems with *unweighted votes* as well. It is worthwhile to explain the similarities and differences between manipulation and control at this example. Betzler et al. [9] describe the unweighted coalitional manipulation problem on p. 55 as follows: “Can one add a certain number of additional votes (called manipulators) to an election such that a distinguished candidate becomes a winner?” This problem is somewhat reminiscent of constructive control by adding voters. However, while the manipulators are free to choose their votes at will, the chair exerting control is confined to a pool of given votes to choose from. Note also that for most voting systems the NP-hardness reductions for control by adding voters tend to be easier than those for control by partition of voters; look, e.g., at the proofs of Theorems 3.19, 3.21, and 3.25 (the latter two using Constructions A.1 and 3.23 and Lemmas A.2 and 3.24).

A different and very insightful direction is taken by Hemaspaandra et al. [47] who analyze whether the complexity of the decision problems modeling the above described attacks on elections is the same as the complexity for the corresponding search problems. In their analysis, they also observed that some pairs of distinctly defined control problems in fact collapse to just one type of control. When defining them formally in Section 2.2, we will mention which control types this comment refers to.

Organization This paper is organized as follows. In Section 2, we recall some notions from social choice theory, define the commonly studied types of control, and explain Bucklin voting and the fallback voting procedure of Brams and Sanver [12] in detail. Our results on the classical and parameterized control complexity of Bucklin voting and fallback voting are presented in Section 3. Finally, Section 4 provides some conclusions and open questions.

³ Hemaspaandra et al. [44] construct, via “hybridization,” a system with perfect control resistance. However, this system is artificial and shouldn’t be used in practice, and it was not designed for that purpose.

⁴ SP-AV is another hybrid system combining approval and preference-based voting; Brams and Sanver [11] proposed the original system and Erdélyi et al. [25] its modification SP-AV. The reason why we said SP-AV is less natural than fallback voting is that, to preserve the votes’ “admissibility” (as required by Brams and Sanver [11] to preclude trivial approval strategies), SP-AV employs an additional rule to (re-)coerce admissibility if in the course of a control action an originally admissible vote becomes inadmissible. As discussed in detail by Baumeister et al. [6], this rule, if applied, changes the approval strategies of the originally cast votes—a severe drawback. In contrast, here we study the *original fallback voting system* of Brams and Sanver [12] where votes, once cast, do not change.

2. Preliminaries

2.1. Elections and voting systems

An *election* (C, V) is given by a finite set C of candidates and a finite list V of votes over C . A *voting system* is a rule that specifies how to determine the winner(s) of any given election. The two voting systems considered in this paper are Bucklin voting and fallback voting. Bucklin voting is named after James W. Bucklin and was used from 1909 till 1922 in Grand Junction, Colorado [49]. Bucklin voting is therefore also referred to as Grand Junction voting. Between 1910 and 1917, it was also adopted in real elections in many other cities in the United States (see, e.g., <http://www.electology.org/bucklin/>); although named “Bucklin voting,” the system they used was actually what Brams and Sanver [12] introduced as fallback voting, a hybrid voting system that combines Bucklin with approval voting.

In *Bucklin voting*, votes are represented as (strict) linear orders over C , i.e., each voter ranks all candidates according to his or her preferences. For example, if $C = \{a, b, c, d\}$ then a vote might look like $c \ d \ a \ b$, i.e., this voter (strictly) prefers c to d , d to a , and a to b . Given an election (C, V) and a candidate $c \in C$, define the *level i score of c in (C, V)* (denoted by $\text{score}_{(C,V)}^i(c)$) as the number of votes in V that rank c among their top i positions. Denoting the *strict majority threshold for a list V of voters* by $\text{maj}(V) = \lfloor \|V\|/2 \rfloor + 1$, the *Bucklin score of c in (C, V)* is the smallest i such that $\text{score}_{(C,V)}^i(c) \geq \text{maj}(V)$. All candidates with the smallest Bucklin score, say k , and the largest level k score are the *Bucklin winners (BV winners, for short) in (C, V)* . If some candidate becomes a Bucklin winner on level k , we call him or her a *level k BV winner in election (C, V)* . Note that a level 1 BV winner must be unique, but there may be more level k BV winners than one for $k > 1$, i.e., an election may have more than one Bucklin winner in general.

In *approval voting*, votes are represented by approval vectors in $\{0, 1\}^{\|C\|}$ (with respect to a fixed order of the candidates in C), where 0 stands for disapproval and 1 stands for approval. Given an election (C, V) and a candidate $c \in C$, define the *approval score of c in (C, V)* (denoted by $\text{score}_{(C,V)}(c)$) as the number of c 's approvals in (C, V) , and all candidates with the largest approval score are the *approval winners in election (C, V)* . Note that an election may have more than one approval winner.

Fallback voting combines Bucklin with approval voting as follows. Each voter provides both an approval vector and a linear ordering of all approved candidates. The subset of candidates approved by a voter is also called his or her *approval strategy*. For simplicity, we will omit the disapproved candidates in each vote.⁵ For example, if $C = \{a, b, c, d\}$ and a voter approves of a, c , and d but disapproves of b , and prefers c to d and d to a , then this vote will be written as: $c \ d \ a$. We will always explicitly state the candidate set, so it will always be clear which candidates participate in an election and which of them are disapproved by which voter (namely those not occurring in his or her vote). Given an election (C, V) and a candidate $c \in C$, the notions of *level i score of c in (C, V)* and *level k fallback voting winner (level k FV winner, for short) in election (C, V)* are defined analogously to the case of Bucklin voting, and if there exists a level k FV winner for some $k \leq \|C\|$, he or she is called a *fallback winner (FV winner, for short) in election (C, V)* . However, unlike in Bucklin voting, in fallback voting it may happen that no candidate reaches a strict majority for any level, due to voters being allowed to disapprove of (any number of) candidates, so it may happen that for no $k \leq \|C\|$ a level k FV winner exists. In such a case, every candidate with the largest (approval) score is an *FV winner in election (C, V)* . Note that Bucklin voting is the special case of fallback voting where each voter approves of all candidates.

As a notation, when a vote contains a subset of the candidate set, such as $c \ D \ a$ for a subset $D \subseteq C$, this is a shorthand for $c \ d_1 \ \dots \ d_\ell \ a$, where the elements of $D = \{d_1, \dots, d_\ell\}$ are ranked with respect to some (tacitly assumed) fixed ordering of all candidates in C . For example, if $C = \{a, b, c, d\}$ is assumed to be ordered lexicographically and $D = \{b, d\}$ then “ $c \ D \ a$ ” is a shorthand for the vote $c \ b \ d \ a$. If the candidate set is downsized in the process of a control action, the deleted candidates are removed in every vote, e.g., a vote $a \ c \ b \ d$ over the candidate set $C = \{a, b, c, d\}$ is altered to $c \ d$ if the subset $C' = \{c, d\}$ of candidates is considered.

2.2. Types of electoral control

We will now formally describe the different types of electoral control, where we will distinguish two variants: In *constructive control* [5], the chair tries to make his or her favorite candidate win; in *destructive control* [43], the chair tries to prevent a despised candidate's victory. We refrain from giving a detailed discussion of natural, real-life scenarios for each of the standard control types that motivate them; these can be found in, e.g., [5,43,34,44,25]. However, we stress that every control type is motivated by an appropriate real-life scenario, and we will briefly point some of them out below.

When we now formally define our standard control types as decision problems, we assume that each election or sub-election in these control problems will be conducted with the voting system at hand (i.e., either Bucklin or fallback voting) and that each vote will be represented as required by the corresponding voting system. We also assume that the chair has complete knowledge of the voters' preferences and/or approval strategies. This assumption may be considered to be unrealistic in certain settings, but is reasonable and natural in certain others, including small-scale elections among humans

⁵ Erdélyi and Rothe [24] use a slightly different notation by separating the approved candidates from the disapproved candidates by a line, where all candidates to the left of this line are ranked and those to its right are unranked.

and even large-scale elections among software agents. More to the point, assuming the chair to have complete information makes sense for our results, as most of our results are NP-hardness lower bounds showing resistance of a voting system against specific control attempts and complexity lower bounds in the complete-information model are inherited by any natural partial-information model; see [43] for a more detailed discussion of this point.

We formally state our control problems in the common instance/question format. We start with the four problems modeling control by adding candidates. In these control scenarios, the chair seeks to make a given preferred candidate win (in the constructive cases) or prevent the victory of a given despised candidate (in the destructive cases) via introducing new candidates from a given pool of spoiler candidates into the election. Faliszewski et al. [34] formalize this problem as follows. (In the definitions of control problems below, whenever we have a proper candidate subset $C' \subset C$ for an election (C, V) then (C', V) denotes the election where the votes in V are restricted to C' .)

CONSTRUCTIVE CONTROL BY ADDING A LIMITED NUMBER OF CANDIDATES (CCAC)

Given:	An election $(C \cup D, V)$, $C \cap D = \emptyset$, a distinguished candidate $c \in C$, and a nonnegative integer k . (C is the set of originally qualified candidates and D is the set of spoiler candidates that may be added.)
Question:	Does there exist a subset $D' \subseteq D$ such that $\ D'\ \leq k$ and c is the unique winner (under the election system at hand) of election $(C \cup D', V)$?

CONSTRUCTIVE CONTROL BY ADDING AN UNLIMITED NUMBER OF CANDIDATES, the problem variant originally proposed by Bartholdi et al. [5], is the same except there is no limit k on the number of spoiler candidates that may be added. We abbreviate this problem variant by CCAUC. Faliszewski et al. [34] discuss in detail the reasons of why it makes sense to also consider the unlimited version of the problem. Although the difference in the definitions may appear to be negligible, note that the complexity of these problems differs significantly in some cases, e.g., in Lull's voting system [34]. The destructive variants of both problems defined above are obtained by asking whether c is *not* a unique winner of election $(C \cup D', V)$. We use the shorthands DCAC and DCAUC.

Control by deleting candidates is defined analogously to control by adding a limited number of candidates, except that the chair now seeks to make a distinguished candidate c win by deleting up to k candidates from the given election.⁶ This control scenario models candidate suppression. For example, by deleting certain candidates other than c the chair may hope that their voters swing to now support c .

CONSTRUCTIVE CONTROL BY DELETING CANDIDATES (CCDC)

Given:	An election (C, V) , a distinguished candidate $c \in C$, and a nonnegative integer k .
Question:	Does there exist a subset $C' \subseteq C$ such that $\ C'\ \leq k$ and c is the unique winner (under the election system at hand) of election $(C - C', V)$?

The destructive version of this problem is the same except that the chair now wants to preclude c from being a unique winner (and, to prevent the problem from being trivial, simply deleting c is not allowed). We use the shorthand DCDC.

Both CONSTRUCTIVE CONTROL BY PARTITION OF CANDIDATES and CONSTRUCTIVE CONTROL BY RUN-OFF PARTITION OF CANDIDATES take as input an election (C, V) and a candidate $c \in C$ and ask whether c can be made the unique winner in a certain two-stage election consisting of one (in the partition case) or two (in the run-off partition case) first-round subelection(s) and a final round. In both variants, following Hemaspaandra et al. [43], we consider two tie-handling rules, TP (“ties promote”) and TE (“ties eliminate”), that enter into force when more candidates than one are tied for winner in any of the first-round subelections:

CONSTRUCTIVE CONTROL BY RUN-OFF PARTITION OF CANDIDATES WITH TP RULE (CCRPC-TP)

Given:	An election (C, V) and a distinguished candidate $c \in C$.
Question:	Is it possible to partition C into C_1 and C_2 such that c is the unique winner (under the election system at hand) of election $(W_1 \cup W_2, V)$, where $W_i, i \in \{1, 2\}$, is the set of winners of subelection (C_i, V) ?

CONSTRUCTIVE CONTROL BY PARTITION OF CANDIDATES WITH TP RULE (CCPC-TP)

Given:	An election (C, V) and a distinguished candidate $c \in C$.
Question:	Is it possible to partition C into C_1 and C_2 such that c is the unique winner (under the election system at hand) of election $(W_1 \cup C_2, V)$, where W_1 is the set of winners of subelection (C_1, V) ?

⁶ No unlimited version has been considered previously for this control type or for the types of control by adding or deleting voters to be defined below.

In both cases, when the TE rule is used, none of multiple, tied first-round subelection winners is promoted to the final round. For example, if we have a run-off and $\|W_2\| \geq 2$ then the final-round election collapses to (W_1, V) ; only a *unique* first-round subelection winner is promoted to the final round in TE. We abbreviate these two problem variants by CCRPC-TE and CCPC-TE. Note that the candidate set in the final round can be empty when the TE rule is used. In this case, the resulting two-stage election has no winner.

It is obvious how to obtain the destructive variants of these four problems formalizing control by candidate partition. We use the shorthands DCRPC-TP, DCPC-TP, DCRPC-TE, and DCPC-TE. Note that in the destructive TE case, a control action is also successful when the winner set is empty (i.e., when there are no participants in the final round of the election due to ties in the subelections).

We have defined all control problems so far in the so-called *unique-winner model*, which is the default in most previous papers on control. By changing the questions in these problems to whether the designated candidate can be made a winner of the resulting election (possibly among several winners) in the constructive cases, or can be prevented from being a winner in the destructive cases, the so-called *nonunique-winner (or co-winner) model* is defined. All our results hold in both models; we will provide detailed proofs in the unique-winner model and (if needed) will explicitly state how to adapt them to also apply to the co-winner case.

These are the originally defined variants of the partition-of-candidates problems due to Hemaspaandra et al. [43]. As mentioned in the introduction, Hemaspaandra et al. [47] recently observed that some pairs of destructive partition-of-candidates cases with the same tie-handling rule actually collapse, depending on the winner model used: DCRPC-TE = DCPC-TE in the unique-winner model, and DCRPC-TP = DCPC-TP and DCRPC-TE = DCPC-TE in the co-winner model, for each election system.

The following example gives a real-life scenario of control by partition of candidates in both cases: with and without a runoff.

Example 2.1. The student representatives of the computer science department want to organize a movie night for the students of their department and it has to be decided which movie to show. All students coming to movie night should have a say in what movie will be watched, so the student representatives organize an election the following way: First, they group the different movies according to categories like “action,” “comedy,” “romance,” “thriller,” and so on, and the students determine the best movies in each category according to their tastes. Then, when the winners in the different categories are known, the students can decide what kind of movie they want to see, so in the final election, they vote over the winning movies from the categories. By limiting the number of categories to two, this situation can be modeled by partition of candidates with runoff.

Another way of organizing this election is the following: To do justice to classic movies that always deserve to be shown, the student’s representatives first choose some classic movies, and then they let the students elect their favorite one among a number of recently released movies. In the final election, the students vote over all the classic movies and the subelection winners among the recent released movies. This models partition of candidates without a runoff.

Other real-life examples include sports tournaments in which certain teams (such as last year’s champion and the team hosting this year’s championship) are given an exemption from qualification.

Turning now to the voter control problems, we start with control by adding voters. This control scenario models attempts by the chair to influence the outcome of elections via introducing new voters. There are many ways of introducing new voters into an election—think, for example, of “get-out-the-vote” drives, or of lowering the age-limit for the right to vote, or of attracting new voters with certain promises or even small gifts.

CONSTRUCTIVE CONTROL BY ADDING VOTERS (CCAV)	
Given:	An election $(C, V \cup V')$, $V \cap V' = \emptyset$, where V is a list of registered voters and V' a pool of as yet unregistered voters that can be added, a distinguished candidate $c \in C$, and a nonnegative integer k .
Question:	Does there exist a sublist $V'' \subseteq V'$ of size at most k such that c is the unique winner (under the election system at hand) of election $(C, V \cup V'')$?

The destructive variant of this problem is the same except that the chair now wants to preclude c from being a unique winner. We use the shorthand DCAV.

Disenfranchisement and other means of voter suppression is modeled as control by deleting voters.

CONSTRUCTIVE CONTROL BY DELETING VOTERS (CCDV)	
Given:	An election (C, V) , a distinguished candidate $c \in C$, and a nonnegative integer k .
Question:	Does there exist a sublist $V' \subseteq V$ such that $\ V'\ \leq k$ and c is the unique winner (under the election system at hand) of election $(C, V - V')$?

Again, the destructive variant of this problem is the same except that the chair now wants to preclude c from being a unique winner. We use the shorthand DCDV.

Also the voter set can be partitioned when controlling an election and we define the corresponding problem using the tie-handling model TP as follows.

CONSTRUCTIVE CONTROL BY PARTITION OF VOTERS (CCPV-TP)	
Given:	An election (C, V) and a distinguished candidate $c \in C$.
Question:	Is it possible to partition V into V_1 and V_2 such that c is the unique winner (under the election system at hand) of election $(W_1 \cup W_2, V)$, where $W_i, i \in \{1, 2\}$, is the set of winners of subelection (C, V_i) ?

The destructive variant of this problem, denoted by DCPV-TP, is defined analogously, except it asks whether c is *not* a unique winner of this two-stage election. In both variants, if one uses the tie-handling model TE instead of TP in the two first-stage subelections, a winner w of subelection (C, V_1) or (C, V_2) proceeds to the final stage if and only if w is the only winner of his or her subelection. We use the shorthands CCPV-TE and DCPV-TE. Each of the four problems just defined models “two-district gerrymandering” (see e.g. [2] for further information on district-gerrymandering in general).

2.3. Classical and parameterized complexity

We assume the reader is familiar with the basic complexity classes such as P and NP. In classical complexity theory, a decision problem A (*polynomial-time many-one*) reduces to a decision problem B if there is a polynomial-time computable function f such that for all inputs x , x is a yes-instance for problem A if and only if $f(x)$ is a yes-instance for problem B . A problem B is NP-hard if every NP problem A reduces to B , and B is NP-complete if it is NP-hard and in NP. A problem B is shown to be NP-hard by exhibiting a reduction to B from a problem A that is already known to be NP-hard. More background on (classical) complexity theory can be found, e.g., in the textbooks by Papadimitriou [59] and Rothe [61].

The theory of parameterized complexity was introduced by Downey and Fellows [18] based on a series of papers in the early 1990s and has developed into a vigorous branch of contemporary computer science, having strong applications in such areas as artificial intelligence and computational biology. The main idea is that for most NP-hard problems, typical inputs have secondary structure beyond the instance size measure n that may significantly affect problem complexity in real-world computing contexts.

As a simple concrete illustration of the issue, the problem ML TYPE CHECKING, concerned with checking the consistency of type declarations in the ML programming language, was noted to be easy to solve in practice, despite being NP-hard. The explanation is that the relevant algorithm runs in time $O(2^k n)$, where n is the instance size (here, the length of the ML program), and k is the secondary measurement: the maximum nesting depth of the type declarations. For real-world ML programs, usually $k \leq 3$ and the algorithm easily solves the problem for typical instances.

This concrete example leads to the general setup of parameterized complexity theory. A *parameterized decision problem* is a decision problem in the classical sense, together with a specification of the secondary measurement (the *parameter*) of interest. The parameter may be an aggregate of several secondary measurements. The central notion is *fixed-parameter tractability* (FPT), meaning solvability in time $f(k)n^c$, where f is an arbitrary function, and c is a fixed constant. One can see that this generalizes polynomial time to this explicitly multivariate (two-dimensional) setting of parameterized decision problems.

There are some parameterized decision problems that seem not to admit fixed-parameter tractable algorithms. The well known graph problem VERTEX COVER, parameterized by the size of a solution set of vertices, is fixed-parameter tractable, with an algorithm having the same running time as in the above ML TYPE CHECKING example: $O(2^k n)$, where here n is the number of vertices in the instance graph, and k is the solution size. In contrast, the graph problem DOMINATING SET, parameterized by the size of a solution set of vertices, seems to admit no algorithm significantly better than $O(n^k)$, based on brute force examination of all up-to- k -subsets of the vertices.

Classical complexity is built from essentially four main ingredients: (1) The central (desirable) notion of polynomial-time complexity. (2) The notion of polynomial-time many-one reducibility that “transmits” the issue of polynomial-time solvability downward in the sense that if a decision problem A reduces to a decision problem B , and if B is polynomial-time solvable, then A is as well. (3) There is a hierarchy of classes of problems that are considered unlikely all to admit polynomial-time algorithms, e.g., NP and beyond; see, e.g., the textbook by Rothe [61] for various hierarchies built upon NP. (4) This consideration is supported by a highly plausible conjecture (namely, that P differs from NP) concerning the difficulty of analyzing the behavior of nondeterministic Turing machines.

Parameterized complexity is similarly structured: (1) The central (desirable) outcome in the two-dimensional setting is fixed-parameter tractability. (2) There is a corresponding notion of reduction between parameterized problems that transmits the issue of FPT solvability downward (see below for the definition). (3) There is a hierarchy of presumably intractable parameterized problem classes; see below and, e.g., the monograph by Downey and Fellows [18]. (4) The presumption is underwritten by a highly plausible conjecture concerning the difficulty of analyzing the behavior of nondeterministic Turing machines, that is a natural (parameterized) variation on the central classical complexity conjecture.

Definition 2.2 (Downey and Fellows [18]). A parameterized decision problem is a language $\mathcal{L} \subseteq \Sigma^* \times \mathbb{N}$. \mathcal{L} is fixed-parameter tractable if there exists some computable function f such that for each input (x, k) of size $n = |(x, k)|$, it can be determined in time $\mathcal{O}(f(k) \cdot n^c)$ whether or not (x, k) is in \mathcal{L} , where c is a constant.

Given two parameterized problems \mathcal{L} and \mathcal{L}' (both encoded over $\Sigma^* \times \mathbb{N}$), we say \mathcal{L} parameterizedly reduces to \mathcal{L}' if there is a function $f : \Sigma^* \times \mathbb{N} \rightarrow \Sigma^* \times \mathbb{N}$ such that for each (x, k) , (1) $f(x, k) = (x', k')$ can be computed in time $\mathcal{O}(g(k) \cdot p(|x|))$ for some function g and some polynomial p , and (2) $(x, k) \in \mathcal{L}$ if and only if $(x', k') \in \mathcal{L}'$, where $k' \leq g(k)$ (that is, k' depends only on k).

A parameterized problem \mathcal{L} is hard for a parameterized complexity class \mathcal{C} if every problem in \mathcal{C} parameterizedly reduces to \mathcal{L} , and \mathcal{L} is complete for \mathcal{C} if it both belongs to \mathcal{C} and is hard for \mathcal{C} .

The main hierarchy of parameterized complexity classes is: $\text{FPT} = \text{W}[0] \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \dots \subseteq \text{W}[t] \subseteq \dots \subseteq \text{XP}$. FPT is the class of fixed-parameter tractable problems. $\text{W}[1]$ is a strong parameterized analogue of NP , as the parameterized k -STEP HALTING PROBLEM FOR NONDETERMINISTIC TURING MACHINES is complete for $\text{W}[1]$ under the above notion of parameterized reducibility [18]. If one can parameterizedly reduce from this problem to some other problem, this other problem is $\text{W}[1]$ -hard.

Many classical decision problems have an obvious parameterized variant for a natural parameter. For example, the CLIQUE problem asks, given a graph G and a positive integer k , whether G has a clique (i.e., a subset of G 's vertices that are pairwise adjacent) of size at least k . In such cases, we let k - Π denote this obvious parameterized variant of the classical decision problem Π . The parameterized k -CLIQUE problem is another problem complete for $\text{W}[1]$, and the parameterized k -DOMINATING SET problem (which will be defined in Section 2.3) is complete for $\text{W}[2]$. The latter two parameterized problems (with parameter k , where k is the solution size) are frequent sources of reductions that show likely parameterized intractability. XP is the class of parameterized decision problems solvable in time $\mathcal{O}(n^{g(k)})$ for some function g . For further background on parameterized complexity theory, see the monographs by Downey and Fellows [18], Niedermeier [57], and Flum and Grohe [39]. Betzler et al. [10] survey a number of FPT and parameterized complexity results in computational social choice, see also the wiki of the parameterized complexity community (fpt.wikidot.com).

We will now introduce the decision problems that will be used for hardness proofs throughout this paper and we begin with the NP -complete problem EXACT COVER BY THREE-SETS, which will be used in Section 3.4 (see, e.g., Garey and Johnson [41]):

EXACT COVER BY THREE-SETS (X3C)	
Given:	A set $B = \{b_1, b_2, \dots, b_{3m}\}$, $m > 1$, and a collection $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ of subsets $S_i \subseteq B$ with $\ S_i\ = 3$ for each i , $1 \leq i \leq n$.
Question:	Is there a subcollection $\mathcal{S}' \subseteq \mathcal{S}$ such that each element of B occurs in exactly one set in \mathcal{S}' ?

Note that X3C is trivial to solve for $m = 1$. For the hardness proofs for several candidate control cases in Section 3.3 and one case of voter control in Section 3.4, we will use a restricted version of the NP -complete problem HITTING SET (see, e.g., Garey and Johnson [41]), which is defined as follows:

RESTRICTED HITTING SET (RHS)	
Given:	A set $B = \{b_1, b_2, \dots, b_m\}$, a collection $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ of nonempty subsets $S_i \subseteq B$ such that $n > m$, and a positive integer k with $1 < k < m$.
Question:	Does \mathcal{S} have a hitting set of size at most k , i.e., is there a set $B' \subseteq B$ with $\ B'\ \leq k$ such that for each i , $S_i \cap B' \neq \emptyset$?

This restriction is needed to ensure that in the election constructed from an RHS instance in Construction A.1 (see p. 657), the scores of certain candidates are bounded. Construction A.1 is an adaptation to Bucklin voting of a construction due to Hemaspaandra et al. [43], namely Construction 4.28 in their paper, which they used to handle several candidate control cases for plurality voting. The election constructed there starts from a differently defined problem that is also called “RESTRICTED HITTING SET” but that is restrictive in another sense by requiring $n(k + 1) \leq m - k$. In both constructions, the restriction serves the same purpose: bounding the scores of certain candidates so as to make the construction work. Observe that our RHS problem is NP -complete as well, i.e., our restriction does not make the problem too easy. For the (straightforward) proof, see the technical report [27].

Regarding parameterized complexity, many $\text{W}[2]$ -hardness results are proven via parameterized reductions from parameterized graph problems. We will prove $\text{W}[2]$ -hardness of certain parameterized control problems via a parameterized reduction from the k -DOMINATING SET problem, which was shown to be $\text{W}[2]$ -complete by Downey and Fellows [18]. Before we introduce this problem, we need some basic graph-theoretic notions.

Definition 2.3. Let $G = (B, A)$ be an undirected graph without loops or multiple edges.⁷ We say that two distinct vertices b_i and b_j are *adjacent* in G if and only if there is an edge $\{b_i, b_j\} \in A$. Adjacent vertices are called *neighbors* in G . The *neighborhood* of a vertex $b_i \in B$ is defined by $N(b_i) = \{b_j \in B \mid \{b_i, b_j\} \in A\}$. The *closed neighborhood* of $b_i \in B$ is defined by $N[b_i] = N(b_i) \cup \{b_i\}$. For a subset $S \subseteq B$, the *neighborhood* of S is defined as $N(S) = \bigcup_{b_i \in S} N(b_i)$ and the *closed neighborhood* of S is defined as $N[S] = \bigcup_{b_i \in S} N[b_i]$.

A subset $B' \subseteq B$ is said to be a *dominating set* in G if for each $b_i \in B - B'$ there is a $b_j \in B'$ such that $\{b_i, b_j\} \in A$. The *size of a dominating set* is the number of its vertices.

Now we can define the $W[2]$ -complete problem k -DOMINATING SET:

k -DOMINATING SET (k -DS)	
Given:	A graph $G = (B, A)$ and a positive integer $k \leq \ B\ $.
Parameter:	k .
Question:	Is there a dominating set of size at most k in G ?

In other words, the (parameterized) dominating set problem tests, given a graph $G = (B, A)$ and an integer k as the parameter, whether there is a subset $B' \subseteq B$ of size at most k such that $B = N[B']$. To distinguish the classical variant of this problem from its parameterized version just defined above, we drop the parameter in the problem name of the latter and simply write DOMINATING SET (DS), one of the standard NP-complete problems (see, e.g., Garey and Johnson [41]). Note that, without loss of generality, we can assume that $n > 2$ holds in any given DS and k -DS instance, respectively, since the thus restricted problem remains NP-complete and $W[2]$ -complete, respectively. The parameterized version k -RHS of RHS can be defined analogously.

Remark 2.4. If Π is a (classical) decision problem in P, then k - Π is in FPT for each parameter k . This gives a useful link between the two easiness notions in classical and parameterized complexity theory. In particular, FPT membership of a parameterized control problem (with respect to any parameter) follows from the voting system's vulnerability to the corresponding control type.

It may be tempting to assume there would be a similarly direct link between the hardness notions in classical and parameterized complexity theory, such as " $W[2]$ -hardness of k - Π immediately implies NP-hardness for Π ." However, this statement is false, in general. The classical framework, which is built around P and reductions, and the parameterized framework, which is built around FPT and parameterized reductions, are essentially orthogonal. For concrete examples, with respect to suitable parameters, VC-DIMENSION (VCD) is unlikely to be NP-hard but k -VCD is complete for $W[1]$, and TOURNAMENT DOMINATING SET (TDS) is unlikely to be NP-hard but k -TDS is $W[2]$ -complete,⁸ see [18] for more details. That being said, if one has a proof of, say, $W[2]$ -hardness by a reduction simultaneously mapping from the NP- and $W[2]$ -complete problems DS and k -DS, respectively, and the transformation is polynomial-time, then of course, the transformation shows both NP- and $W[2]$ -hardness at the same time, and we will use this fact in some of our proofs.

In the reductions presented in Sections 3.3 and 3.4, we will always start from a given instance of an NP-complete and/or $W[2]$ -complete decision problem suitable for the control problem at hand, that is, we will always start from a given X3C, RHS, DS, or k -DS instance. In the upcoming constructions, the set of candidates in the elections to be defined will always contain the set B from these instances. That is, for each element (or vertex) $b_i \in B$ there is a candidate b_i in the election constructed, and it will always be clear from the context whether we mean an element (or a vertex) or a candidate when writing b_i . We will also refer to S_i (or $N[b_i]$) as a "subset of the candidates," namely, the set of candidates corresponding to the elements (or vertices) in B that are in S_i (or in vertex b_i 's closed neighborhood, $N[b_i]$, $1 \leq i \leq n$). (Recall the sets S_i from the definition of X3C and RHS and the notion of $N[b_i]$ from Definition 2.3.)

2.4. Immunity, susceptibility, resistance, and vulnerability

Let $\mathcal{C}\mathfrak{T}$ be a control type (e.g., $\mathcal{C}\mathfrak{T}$ might stand for "constructive control by partition of voters in model TP" or any of the other types of control defined in Section 2.2).

Definition 2.5. We say a voting system is *immune* to $\mathcal{C}\mathfrak{T}$ if it is impossible for the chair to make the given candidate a unique winner in the constructive case (not a unique winner in the destructive case) via exerting control of type $\mathcal{C}\mathfrak{T}$, i.e., for

⁷ We denote the vertex set of a graph not by V , as would be common, but rather by B , in order to avoid confusion with voter lists (for which "V" is reserved in this paper) and also in accordance with B being the base set in the instances of the problems X3C and RHS.

⁸ TDS is the problem of deciding whether a given tournament has a dominating set of size at most k (the parameter in k -TDS). VCD is related to the so-called "Vapnik–Chervonenkis dimension" (which is the parameter in k -VCD), a problem quite central in learning theory; see [18] for the definition and more details.

Table 1

Overview of classical and parameterized complexity results for control in Bucklin and fallback voting. All results hold in both the co-winner and the unique-winner model. Key: I = immune, S = susceptible, R = resistant, R* = parameterizedly resistant, V = vulnerable, TE = ties eliminate, and TP = ties promote. Results new to this paper are in boldface. Results for approval voting are due to Hemaspaandra et al. [43]. Results for SP-AV are due to Erdélyi et al. [25].

Control by	Fallback Voting		Bucklin Voting		SP-AV		Approval Voting	
	Const.	Dest.	Const.	Dest.	Const.	Dest.	Const.	Dest.
Adding Candidates	R*	R*	R*	R*	R	R	I	V
Adding Candidates unlimited	R	R	R	R	R	R	I	V
Deleting Candidates	R*	R*	R*	R*	R	R	V	I
Partition of Candidates – TE	R		R		R		V	
Run-off Partition of Candidates – TE	R	R	R	R	R	R	V	I
Partition of Candidates – TP	R		R		R		I	
Run-off Partition of Candidates – TP	R	R	R	R	R	R	I	I
Adding Voters	R*	V	R*	V	R	V	R	V
Deleting Voters	R*	V	R*	V	R	V	R	V
Partition of Voters – TE	R	R	R	R	R	V	R	V
Partition of Voters – TP	R	R	R	S	R	R	R	V

any given election and any given distinguished candidate, there is no successful control action of type $\mathcal{C}\mathfrak{I}$. We say a voting system is *susceptible to $\mathcal{C}\mathfrak{I}$* if it is not immune to $\mathcal{C}\mathfrak{I}$. A voting system that is susceptible to $\mathcal{C}\mathfrak{I}$ is said to be *vulnerable to $\mathcal{C}\mathfrak{I}$* if the control problem corresponding to $\mathcal{C}\mathfrak{I}$ can be solved in polynomial time, and *resistant to $\mathcal{C}\mathfrak{I}$* if the control problem corresponding to $\mathcal{C}\mathfrak{I}$ is NP-hard.

These notions are due to Bartholdi et al. [5] (except that we follow the now more common approach of Hemaspaandra et al. [44] who define *resistant* to mean “susceptible and NP-hard” rather than “susceptible and NP-complete”). In analogy to the classical complexity notion, we say that a voting system is *parameterizedly resistant to $\mathcal{C}\mathfrak{I}$* (with respect to a specified parameter) if it is susceptible to $\mathcal{C}\mathfrak{I}$ and the parameterized decision problem corresponding to $\mathcal{C}\mathfrak{I}$ and this parameter is W[1]-hard. For example, a natural parameter to look at for control by deleting candidates is the number of candidates deleted. Note that W[t]-hardness for $t > 1$ implies W[1]-hardness, so this definition captures hardness for the W-hierarchy.

3. Control complexity in Bucklin and fallback voting

3.1. Overview

Table 1 shows in boldface our results on the control complexity of Bucklin voting and fallback voting for all standard control types. Recall from Remark 2.4 that for every R* entry in the table (i.e., for every parameterized resistance w.r.t. the parameters stated later in our theorems and corollaries), we will show a corresponding resistance (i.e., NP-hardness) result by essentially the same reduction. Since fallback voting combines Bucklin and approval voting, the table also shows the results for approval voting due to Hemaspaandra et al. [43],⁹ for comparison. The other voting system displayed in the table, SP-AV, is yet another hybrid voting system combining approval with preference-based voting, which has been introduced by Brams and Sanver [11] and modified and studied by Erdélyi et al. [25]. Prior to the present work, SP-AV has been the system with the most known resistances to electoral control among natural voting systems with polynomial-time winner determination. As argued in the introduction, both fallback voting and Bucklin voting are much more natural voting systems than SP-AV as modified by Erdélyi et al. [25].

Table 2 gives an overview of the reductions used to prove the new (i.e., boldfaced) resistance and parameterized resistance results in Table 1. The first column states the problem from which we reduce, and the second column the control problem shown to be (parameterizedly) resistant. The problem name has the prefix “BV” if this control problem refers to Bucklin voting (which immediately implies hardness also for fallback voting). The only problem name with the prefix “FV” for fallback voting, FV-DCPV-TP in the last row, refers to the case where the complexity is open for Bucklin voting. The third column of Table 2 points at the corresponding theorem or construction and the fourth column states whether it is a parameterized reduction or not.

3.2. Susceptibility

To show that a voting system is susceptible to a certain type of control, it is enough to construct a suitable example of an election where control of this type is possible. For all types of control considered here, such an example can easily be found for Bucklin elections (see [27]).

Lemma 3.1. *Bucklin voting is susceptible to each control type defined in Section 2.2, in both winner models.*

⁹ Note that the results for control by adding a limited number of candidates in approval voting, though not explicitly considered by Hemaspaandra et al. [43], can be obtained straightforwardly from their proofs for the corresponding “unlimited” adding-candidates case.

Table 2
Overview of the reductions used to prove the new results in Table 1.

Reduction from	to	Reference	parameterized?
k-DS	BV-CCDC	Theorem 3.3	yes
	BV-DCDC	Theorem 3.4	
	BV-CCAV	Theorem 3.11	
	BV-CCDV	Theorem 3.15	
	BV-DCPV-TE	Construction A.1 and Theorem 3.21	no
RHS	BV-CCAC	Construction 3.6 and Theorem 3.8	yes
	BV-DCAC		
	BV-CCAUC		
	BV-DCAUC		
	BV-CCPC-TE	Construction 3.6 and Theorem 3.9	no
	BV-CCPC-TP		
	BV-DCPC-TE		
	BV-DCPC-TP		
	BV-CCRPC-TE		
	BV-CCRPC-TP		
	FV-DCPV-TP	Construction 3.23 and Theorem 3.25	
X3C	BV-CCPV-TE	Theorem 3.19	
	BV-CCPV-TP		

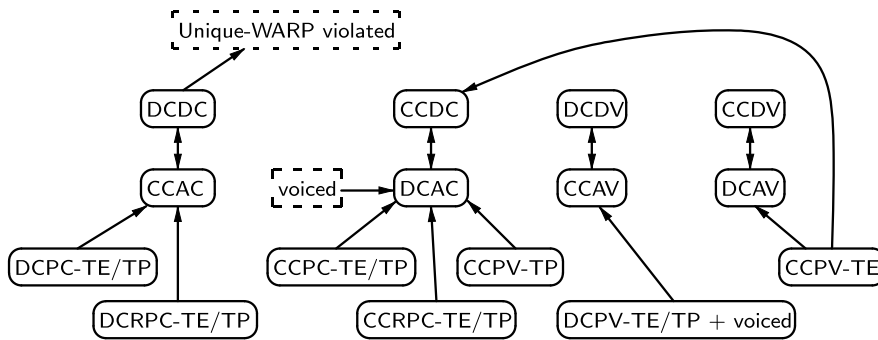


Fig. 1. Susceptibility links in various control scenarios due to Hemaspaandra et al. [43].

Since Bucklin voting is a special case of fallback voting, fallback voting is also susceptible to all common types of control.

Corollary 3.2. Fallback voting is susceptible to each control type defined in Section 2.2, in both winner models.

Our proofs which can be found in detail in the technical report of Erdélyi et al. [27] do not provide one example for each type of control but make use of the results of Hemaspaandra et al. [43] that provide general proofs of and links between certain susceptibility cases. For the sake of self-containment, Fig. 1, which is taken from [63, p. 199], gives an overview of the susceptibility links in various control scenarios from Hemaspaandra et al. [43]. An arrow between two control types in this figure, say $\mathcal{C}_1 \rightarrow \mathcal{C}_2$, means that susceptibility to \mathcal{C}_1 implies susceptibility to \mathcal{C}_2 ; and the arrows between control types and the two properties in dashed boxes mean that (1) susceptibility to destructive control by deleting candidates implies that Unique-WARP is violated,¹⁰ and (2) every voiced¹¹ election system is susceptible to destructive control by adding candidates.

3.3. Candidate control

In this section, we show the (parameterized) resistance results for candidate control in Bucklin and fallback voting. Recall from Section 2.4 that “parameterized resistance” (indicated by the R^* entries in Table 1) refers to $W[2]$ -hardness of the corresponding control problem with respect to a specified parameter, and “resistance” (indicated by the R entries in Table 1) refers to NP-hardness of the corresponding control problem. Also, recall from Section 2.3 that in the reductions to be presented in this section, the elections constructed will always contain a subset B of candidates, where each candidate

¹⁰ The unique variant of the “weak axiom of revealed preferences” (WARP) is violated if there is a unique winner of an election who is not the unique winner of some subelection containing him or her.

¹¹ An election system is said to be *voiced* if the single candidate in any one-candidate election always wins.

Table 3

Level i scores in (C, V) for $i \in \{k+1, k+2, n+k, n+k+1\}$ and the candidates in $C - (X \cup Y)$.

	$b_i \in B$	w	$d_j \in D$
$score^{k+1}$	$\leq n$	0	$n - k$
$score^{k+2}$	$\leq n$	1	$\leq n - 1$
$score^{n+k}$	$\leq n$	1	n
$score^{n+k+1}$	$\leq n$	$n + 1$	$n + 1$

b_i corresponds to the element b_i from the set B given in the (parameterized) decision problem the reduction starts from. It will always be clear from the context whether a candidate or an element is meant by b_i .

Theorem 3.3. *Bucklin voting is resistant to constructive control by deleting candidates, and is parameterizedly resistant when this control problem is parameterized by the number of candidates deleted. Each result holds in both winner models.*

Proof. In light of Remark 2.4 it is enough to give just one reduction to prove both claims at the same time. Let (G, k) with $G = (B, E)$ be a given instance of k -DOMINATING SET as described in Section 2.3. Without loss of generality, we may assume that $k < n = \|B\|$, since the set B of all vertices is a trivial dominating set in G . We start with the unique-winner model.

Define the election (C, V) , where $C = B \cup D \cup \{w\} \cup X \cup Y$ is the set of candidates, w is the distinguished candidate, D is a set of “co-winners” (see below), and X and Y are sets of padding candidates.¹²

Co-winners in D : D is a set of $k + 1$ candidates that tie with w . These candidates prevent that deleting up to k co-winners of election (C, V) makes w the unique winner.

Padding candidates in X : X is a set of $n(n + k) - \sum_{i=1}^n \|N[b_i]\|$ candidates such that for each $i, 1 \leq i \leq n$, we can find a subset $X_i \subseteq X$ with $n + k - \|N[b_i]\|$ elements such that $X_r \cap X_s = \emptyset$ for all $r, s \in \{1, \dots, n\}$ with $r \neq s$. These subsets ensure that w is always placed at the $(n + k + 1)$ st position in the first voter group of V below.

Padding candidates in Y : Y is a set of $n(k + 1)$ candidates such that for each $j, 1 \leq j \leq k + 1$, we can find a subset $Y_j \subseteq Y$ with n elements such that $Y_r \cap Y_s = \emptyset$ for all $r, s \in \{1, \dots, k + 1\}$ with $r \neq s$. These subsets ensure that each $d_j \in D$ is always placed at the $(n + k + 1)$ st position in the second voter group of V below.

V is the following collection of $2n + 1$ voters, so that we have a strict majority with $n + 1$ votes:

#	For each ...	number of votes	ranking of candidates
1	$i \in \{1, \dots, n\}$	1	$N[b_i] \ X_i \ w \ ((B - N[b_i]) \cup (X - X_i) \cup Y) \ D$
2	$j \in \{1, \dots, k + 1\}$	1	$Y_j \ (D - \{d_j\}) \ d_j \ (B \cup X \cup (Y - Y_j) \cup \{w\})$
3		$n - k - 1$	$D \ (X \cup Y) \ w \ B$
4		1	$D \ w \ (X \cup Y) \ B$

Note that when up to k candidates are deleted (no matter which ones), the candidates from D can never be among the top $n + k$ candidates in the votes of the first voter group. Table 3 shows the scores on the relevant levels of the relevant candidates in election (C, V) .

Note that the candidates in D and candidate w are the only level $n + k + 1$ Bucklin winners of election (C, V) , since there is no other candidate reaching a strict majority of $n + 1$ votes or more on any level up to $n + k + 1$.

We claim that G has a dominating set of size k if and only if w can be made the unique Bucklin winner by deleting at most k candidates.

From left to right: Suppose G has a dominating set $B' \subseteq B$ of size k . Delete the corresponding candidates from C . Since B' is a dominating set in G (i.e., $B = N[B']$), every $b_i \in B$ has a neighbor in B' or is itself in B' , which means that in election $(C - B', V)$ candidate w gets pushed at least one position to the left in each of the n votes in the first voter group. So w reaches a strict majority already on level $n + k$ with a score of $n + 1$. Since no other candidate does so (in particular, no candidate in D), it follows that w is the unique level $n + k$ Bucklin winner of election $(C - B', V)$.

From right to left: Suppose w can be made the unique Bucklin winner of the election by deleting at most k candidates. Since there are $k + 1$ candidates other than w (namely, those in D) having a strict majority on level $n + k + 1$ in election (C, V) , deleting k candidates from D is not sufficient for making w the unique Bucklin winner of the resulting election. So by deleting at most k candidates, w must become the unique Bucklin winner on a level lower than or equal to $n + k$. This is possible only if w is pushed at least one position to the left in all votes from the first voter group. This, however, implies that the $k' \leq k$ deleted candidates either are all contained in B and correspond to a dominating set of size k' for G , or are in $B \cup X$. Note that not all deleted candidates can be contained in X , since $k < n$ and the sets $X_i, 1 \leq i \leq n$, are pairwise

¹² Note that in this construction as well as in later constructions, the subsets of padding candidates are always constructed so as to ensure that, at least up to a certain level, no padding candidate scores enough points to be relevant for the outcome of the election. So in the following argument the padding candidates are mainly ignored and their scores are not listed in the overview tables.

disjoint. If some of the k' deleted candidates are in X , say $\ell < k'$ of them, let B' be the set containing the $k' - \ell$ other candidates that have been deleted. For each i , $1 \leq i \leq n$, if in the i th voter of the first group no candidate from $N[b_i]$ was deleted but a candidate x_j from X_i , add an arbitrary candidate from $N[b_i]$ to B' instead of x_j . This yields again a dominating set of size k' for G . In both cases, if $k' < k$ then by adding $k - k'$ further candidates from B (which is possible due to $k < n$) we obtain a dominating set of size k for G .

Note that this polynomial-time reduction is parameterized, as the given parameter k of k -DOMINATING SET is the same parameter k that bounds the number of candidates allowed to be deleted in the control problem.

For showing resistance in the co-winner model, the definition of the padding candidates in X has to be changed such that in each voter of the first group one more padding candidate is ranked ahead of w . This ensures that w is not a winner in the election (C, V) and, furthermore, reaches a strict majority one level later than the candidates in D . Thus, the remaining argumentation can be adapted straightforwardly. \square

The proof of the following theorem is similar to that of [Theorem 3.3](#) and is deferred to [Appendix A](#).

Theorem 3.4. *Bucklin voting is resistant to destructive control by deleting candidates, and is parameterizedly resistant when this control problem is parameterized by the number of candidates deleted. Each result holds in both winner models.*

Corollary 3.5. *Fallback voting is resistant to constructive and destructive control by deleting candidates, and is parameterizedly resistant when these two control problems are parameterized by the number of candidates deleted. Each result holds in both winner models.*

[Construction 3.6](#) will be applied to prove the remaining ten cases of candidate control, establishing resistance in each case. For some of the results, parameterized resistance can be established, as well. This construction adapts [Construction 4.28](#) of [Hemaspaandra et al. \[43\]](#), which they used to handle certain candidate control cases for plurality voting.¹³ Note that they start from a HITTING SET instance, while we reduce from the restricted version RHS of this problem defined in [Section 2.3](#), and in our construction there is one additional candidate, d , and there are two more voter groups than in their construction. These modifications are needed because Bucklin voting is based on a more involved procedure than plurality voting.

Construction 3.6. *Let (B, S, k) be an instance of RHS, with $B = \{b_1, b_2, \dots, b_m\}$ a set, $S = \{S_1, S_2, \dots, S_n\}$ a collection of nonempty subsets $S_i \subseteq B$ such that $n > m$, and $k < m$ a positive integer. (Thus, $n > m > k > 1$.)*

Define the election (C, V) , where $C = B \cup \{c, d, w\}$ is the candidate set and where V consists of the following $6n(k+1) + 4m + 11$ voters:

#	For each ...	number of voters	ranking of candidates
1		$2m + 1$	$c \ d \ B \ w$
2		$2n + 2k(n - 1) + 3$	$c \ w \ d \ B$
3		$2n(k + 1) + 5$	$w \ c \ d \ B$
4	$i \in \{1, \dots, n\}$	$2(k + 1)$	$d \ S_i \ c \ w \ (B - S_i)$
5	$j \in \{1, \dots, m\}$	2	$d \ b_j \ w \ c \ (B - \{b_j\})$
6		$2(k + 1)$	$d \ w \ c \ B$

We will show the following lemma that is used to establish the upcoming results.

Lemma 3.7. *Consider the election (C, V) constructed according to [Construction 3.6](#) from a RHS instance (B, S, k) .*

1. c is the unique level 2 BV winner of election $(\{c, d, w\}, V)$.
2. If S has a hitting set B' of size k , then w is the unique BV winner of election $(B' \cup \{c, d, w\}, V)$.
3. Let $D \subseteq B \cup \{d, w\}$. If c is not a unique BV winner of election $(D \cup \{c\}, V)$, then there exists a set $B' \subseteq B$ such that
 - (a) $D = B' \cup \{d, w\}$,
 - (b) w is a level 2 BV winner of election $(B' \cup \{c, d, w\}, V)$, and
 - (c) B' is a hitting set for S of size at most k .

Proof. For the first part, note that there is no level 1 BV winner in election $(\{c, d, w\}, V)$ and we have the following level 2 scores in this election:

$$\text{score}_{(\{c,d,w\},V)}^2(c) = 6n(k+1) + 2(m-k) + 9,$$

¹³ Their construction was also useful in the proofs of most candidate control results for SP-AV [\[25\]](#), so the structure of the constructions and the arguments in the proofs of [Lemma 3.7](#) and [Theorem 3.9](#) are adaptations of those by [Hemaspaandra et al. \[43\]](#) and [Erdélyi et al. \[25\]](#), tailored here to Bucklin voting.

$$\text{score}_{(\{c,d,w\},V)}^2(d) = 2n(k+1) + 4m + 2k + 3,$$

$$\text{score}_{(\{c,d,w\},V)}^2(w) = 4n(k+1) + 2m + 10.$$

Since $n > m$ (which implies $n > k$), we have:

$$\text{score}_{(\{c,d,w\},V)}^2(c) - \text{score}_{(\{c,d,w\},V)}^2(d) = 4n(k+1) - (2m + 4k) + 6 > 0,$$

$$\text{score}_{(\{c,d,w\},V)}^2(c) - \text{score}_{(\{c,d,w\},V)}^2(w) = 2n(k+1) - (2k + 1) > 0.$$

Thus, c is the unique level 2 BV winner of election $(\{c, d, w\}, V)$.

For the second part, suppose that B' is a hitting set for \mathcal{S} of size k . Then there is no level 1 BV winner in election $(B' \cup \{c, d, w\}, V)$, and we have the following level 2 scores:

$$\text{score}_{(B' \cup \{c,d,w\},V)}^2(c) = 4n(k+1) + 2(m-k) + 9,$$

$$\text{score}_{(B' \cup \{c,d,w\},V)}^2(d) = 2n(k+1) + 4m + 2k + 3,$$

$$\text{score}_{(B' \cup \{c,d,w\},V)}^2(w) = 4n(k+1) + 2(m-k) + 10,$$

$$\text{score}_{(B' \cup \{c,d,w\},V)}^2(b_j) \leq 2n(k+1) + 2 \quad \text{for all } b_j \in B'.$$

It follows that w is the unique level 2 BV winner of election $(B' \cup \{c, d, w\}, V)$.

For the third part, let $D \subseteq B \cup \{d, w\}$. Suppose c is not a unique BV winner of election $(D \cup \{c\}, V)$.

(3a) Besides c , only w has a strict majority of votes on the second level and only w can tie or beat c in $(D \cup \{c\}, V)$. Thus, since c is not a unique BV winner of election $(D \cup \{c\}, V)$, w is clearly in D . In $(D \cup \{c\}, V)$, candidate w has no level 1 strict majority and candidate c has already on level 2 a strict majority. Thus, w must tie or beat c on level 2. For a contradiction, suppose $d \notin D$. Then

$$\text{score}_{(D \cup \{c\},V)}^2(c) \geq 4n(k+1) + 2m + 11;$$

$$\text{score}_{(D \cup \{c\},V)}^2(w) = 4n(k+1) + 2m + 10,$$

which contradicts the above observation that w ties with or beats c on level 2. Thus, $D = B' \cup \{d, w\}$, where $B' \subseteq B$.

(3b) This part follows immediately from the proof of part (3a).

(3c) Let ℓ be the number of sets in \mathcal{S} not hit by B' . We have that

$$\text{score}_{(B' \cup \{c,d,w\},V)}^2(w) = 4n(k+1) + 10 + 2(m - \|B'\|),$$

$$\text{score}_{(B' \cup \{c,d,w\},V)}^2(c) = 2(m-k) + 4n(k+1) + 9 + 2(k+1)\ell.$$

From part (3b) we know that $\text{score}_{(B' \cup \{c,d,w\},V)}^2(w) \geq \text{score}_{(B' \cup \{c,d,w\},V)}^2(c)$, so

$$4n(k+1) + 10 + 2(m - \|B'\|) \geq 2(m-k) + 4n(k+1) + 9 + 2(k+1)\ell.$$

The above inequality implies $1 > \frac{1}{2} \geq \|B'\| - k + (k+1)\ell$.

Since $T = \|B'\| - k + (k+1)\ell$ is an integer, we have $T \leq 0$. If $T = 0$ then $\ell = 0$ and $\|B'\| = k$. Now assume $T < 0$. If $\ell = 0$, B' is a hitting set with $\|B'\| < k$, and if $\ell > 0$ then $(k+1)\ell > k$, which contradicts $T = \|B'\| - k + (k+1)\ell < 0$. In each possible case, we have a hitting set (as $\ell = 0$) of size at most k .

Note that the third part also holds when c is not a winner in election $(D \cup \{c\})$: Parts (3a) and (3b) follow immediately while in part (3c) the only change is that $\text{score}_{(B' \cup \{c,d,w\},V)}^2(w) > \text{score}_{(B' \cup \{c,d,w\},V)}^2(c)$ which directly leads to case of $T = 0$. \square

Now we are ready to handle the eight remaining cases of candidate control.

Theorem 3.8. *Bucklin voting is resistant to constructive and destructive control by adding an unlimited number of candidates. Furthermore, Bucklin voting is resistant to constructive and destructive control by adding a limited number of candidates and is parameterizedly resistant when these two control problems are parameterized by the number of candidates added. Each result holds in both winner models.*

Proof. Susceptibility holds by Lemma 3.1. NP-hardness follows immediately from Lemma 3.7, via mapping the RHS instance (B, \mathcal{S}, k) either to the instance $((\{c, d, w\} \cup B, V), w)$ of CCAUC, or to the instance $((\{c, d, w\} \cup B, V), c)$ of DCAUC, or to the instance $((\{c, d, w\} \cup B, V), k, w)$ of CCAC, or to the instance $((\{c, d, w\} \cup B, V), k, c)$ of DCAC, where in each case c, d , and w are the qualified candidates and B is the set of spoiler candidates. \square

Theorem 3.9. *Bucklin voting is resistant to constructive and destructive control by partition of candidates and by run-off partition of candidates (for each in both tie-handling models, TE and TP). Each result holds in both winner models.*

Proof. Susceptibility holds by Lemma 3.1, so it remains to show NP-hardness. For the constructive cases, map the given RHS instance (B, \mathcal{S}, k) to the election (C, V) from Construction 3.6 with w being the distinguished candidate. We claim that \mathcal{S} has a hitting set of size at most k if and only if w can be made the unique BV winner by exerting control via any of our four control scenarios (partition of candidates with or without run-off, and for each in either tie-handling model, TE and TP).

From left to right: Suppose \mathcal{S} has a hitting set $B' \subseteq B$ of size k . Partition the set of candidates into the two subsets $C_1 = B' \cup \{c, d, w\}$ and $C_2 = C - C_1$. According to Lemma 3.7, w is the unique level 2 BV winner of subelection $(C_1, V) = (B' \cup \{c, d, w\}, V)$. No matter whether we have a run-off or not, and regardless of the tie-handling rule used, the opponents of w in the final stage (if there are any opponents at all) each are candidates from B . Since $n > m$, w has a majority in the final stage on the first level with a score of $4n(k+1) + 9$. Thus, w is the unique BV winner of the resulting election.

From right to left: Suppose w can be made the unique BV winner via any of our four control scenarios. Since c is not a BV winner of the election, there is a subset $D \subseteq B \cup \{d, w\}$ of candidates such that c is not a unique BV winner of election $(D \cup \{c\}, V)$. By Lemma 3.7, there exists a hitting set for \mathcal{S} of size at most k .

For the four destructive cases, we simply change the roles of c and w in the above argument.

The above proof works both in the unique-winner and the co-winner model. \square

Corollary 3.10. *Fallback voting is resistant to constructive and destructive control by partition of candidates and by run-off partition of candidates (for each in both tie-handling models, TE and TP). Each result holds in both winner models.*

3.4. Voter control

Turning now to voter control for Bucklin voting and fallback voting, we start with control by adding and deleting voters, where we have both (classical) resistance and parameterized resistance results. As in the previous section, recall that “parameterized resistance” (the R^* entries in Table 1) refers to $W[2]$ -hardness of the corresponding control problem with respect to a specified parameter, and that “resistance” (the R entries in Table 1) refers to NP-hardness of the corresponding control problem. Again, recall from Section 2.3 that in the reductions to be presented in this section, the elections constructed will always contain a subset B of candidates, where each candidate b_i corresponds to the element b_i from the set B given in the (parameterized) decision problem the reduction starts from, and it will always be clear from the context whether b_i is meant to be a candidate or an element.

Theorem 3.11. *Bucklin voting is resistant to constructive control by adding voters, and is parameterizedly resistant when this control problem is parameterized by the number of voters added. Each result holds in both winner models.*

Proof. Again, in light of Remark 2.4 it is enough to give one reduction that proves both claims at the same time. Let (G, k) with $G = (B, E)$ be a given instance of k -DOMINATING SET as described above. Define the election $(C, V \cup U)$, where $C = \{c, w\} \cup B \cup X \cup Y$ is the set of candidates, w is the distinguished candidate, and X and Y are sets of padding candidates (recall Footnote 12).

Padding candidates in X : X is a set of $\sum_{i=1}^n \|N[b_i]\|$ candidates such that for each i , $1 \leq i \leq n$, we can find a subset $X_i \subseteq X$ with $\|N[b_i]\|$ elements such that $X_r \cap X_s = \emptyset$ for all $r, s \in \{1, \dots, n\}$ with $r \neq s$. These subsets ensure that w is always placed at the $(n+1)$ st position in the votes of the unregistered voters in U below.

Padding candidates in Y : Y is a set of n padding candidates ensuring that none of the candidates in B is ranked among the first n candidates in the votes of the registered voters in V below.

V is the collection of registered and U is the collection of unregistered voters. $V \cup U$ consists of the following $n+k-1$ voters:

Voter list	For each ...	number of voters	ranking of candidates
V		$k-1$	$c \ Y \ B \ w \ X$
U	$i \in \{1, \dots, n\}$	1	$(B - N[b_i]) \ X_i \ w \ c \ (N[b_i] \cup (X - X_i) \cup Y)$

Clearly, c is the unique level 1 Bucklin winner of election (C, V) .

We claim that G has a dominating set of size k if and only if w can be made the unique Bucklin winner by adding at most k voters from U .

From left to right: Suppose G has a dominating set B' of size k . Add the corresponding voters from U to the election (i.e., each voter u_i for which $b_i \in B'$). Now we have an election with $2k-1$ voters, so the strict majority threshold is k . Since B' is a dominating set, we have $B = N[B']$, so for each $b_j \in B$ there is at least one of the added voters u_i such that

$b_j \in N[b_i]$, which means that b_j is ranked to the right of w in these k added votes. It follows that up to level $n + 1$ only candidate w will reach this threshold of k , hence w is the unique Bucklin winner of this election.

From right to left: Suppose w can be made the unique Bucklin winner by adding at most k voters from U . Denote the set of these voters by U' and note that $\|U'\| \leq k$. Note further that $score_{(C, V \cup U')}^1(c) = score_{(C, V)}^1(c) = k - 1$, that is, c reaches a score of $k - 1$ already on the first level (with or without adding U'). However, if any candidate has a strict majority already on the first level, then he or she is the unique Bucklin winner of the election. As w is the unique Bucklin winner of election $(C, V \cup U')$, the strict majority threshold for $V \cup U'$ must be greater than $k - 1$. This, in turn, implies $\|U'\| \geq k$, so $\|U'\| = k$ and the strict majority threshold for $V \cup U'$ is exactly k . Note that $score_{(C, V \cup U')}^{n+1}(w) = k > k - 1 = score_{(C, V \cup U')}^{n+1}(x)$ and $score_{(C, V \cup U')}^n(w) = 0$. Moreover, since adding the voters from U' to the election has made w the unique Bucklin winner of election $(C, V \cup U')$, none of the candidates in B can be ranked among the first n candidates by each voter in U' ; otherwise (i.e., if some candidate $b_j \in B$ would be ranked among the first n candidates by each voter in U'), we would have $score_{(C, V \cup U')}^n(b_j) = k$, i.e., b_j would reach a strict majority in $(C, V \cup U')$ earlier than w , a contradiction. But this means that the voters in U' correspond to a dominating set of size k in G .

Note that this polynomial-time reduction is parameterized, as the given parameter k of k -DOMINATING SET is the same parameter k that bounds the number of voters allowed to be added in this control problem.

The above proof works both in the unique-winner and the co-winner model. \square

Corollary 3.12. *Fallback voting is resistant to constructive control by adding voters, and is parameterizedly resistant when this control problem is parameterized by the number of voters added. Each result holds in both winner models.*

In contrast to Corollary 3.12, fallback voting is vulnerable to destructive control by adding voters and, as we will see later in Theorem 3.17, it is also vulnerable to destructive control by deleting voters. In fact, the proof of Theorem 3.13 shows something slightly stronger: Fallback voting is what Hemaspaandra et al. [43] call “certifiably vulnerable” to this type of destructive voter control, i.e., the algorithm showing vulnerability to destructive control by adding voters not only decides whether or not control is possible but it even computes a successful control action if one exists. Note that in the proof of Theorem 3.13, after giving a high-level description of the algorithm, we present the algorithm in detail and argue for its correctness as we go. Therefore, we refrain from giving a formal presentation (e.g., in pseudocode, which would be a bit cumbersome to read). Note further that the algorithm is not designed so as to optimizing its runtime (as then it might be harder to see it is correct); rather, we focus on clarity regarding the arguments for correctness. Thus, we give an upper bound for the runtime (with respect to the number of candidates in the given election) that still might be improved.

Theorem 3.13. *Fallback voting is vulnerable to destructive control by adding voters in both winner models, via algorithms each having a worst-case runtime in $\mathcal{O}(m^2(n' + p(n + n')))$, where m is the number of candidates, n (respectively, n') is the number of votes in V (respectively, in V'), and $p(x)$ is a polynomial giving an upper bound for the time needed to compute the level scores in an election with x voters.*

Proof. Susceptibility holds by Corollary 3.2. We present a polynomial-time algorithm for solving the destructive control by adding voters case. We will make use of the following notation. Given an election (C, V) , recall that $maj(V) = \lfloor \|V\|/2 \rfloor + 1$ denotes the strict majority threshold for V , and define the deficit of candidate $d \in C$ for reaching a strict majority in (C, V) on level i , $1 \leq i \leq \|C\|$, by

$$def_{(C, V)}^i(d) = maj(V) - score_{(C, V)}^i(d).$$

Similarly, define the surplus of candidate $d \in C$ over a strict majority in (C, V) on level i , $1 \leq i \leq \|C\|$, by

$$surplus_{(C, V)}^i(d) = score_{(C, V)}^i(d) - maj(V) = -def_{(C, V)}^i(d).$$

The input to our algorithm is an election $(C, V \cup V')$ (where C is the set of candidates, V is the collection of registered voters, and V' is the collection of unregistered voters), a distinguished candidate $c \in C$, and an integer ℓ (the number of voters allowed to be added). The algorithm either outputs a sublist $V'' \subseteq V'$, $\|V''\| \leq \ell$, that describes a successful control action (if any exists), or indicates that control is impossible for this input.

We give a high-level description of the algorithm. We assume that c is initially the unique FV winner of election (C, V) ; otherwise, the algorithm simply outputs $V'' = \emptyset$ and halts, since there is no need to add any voters from V' . Let k be the largest number of candidates any voter in $V \cup V'$ approves of. Clearly, $k \leq \|C\|$. The algorithm proceeds in at most $k + 1$ stages, where the last stage is the *approval stage* that checks whether c can be dethroned as a unique FV winner by approval score via adding at most ℓ voters from V' , and all preceding stages are *majority stages* that check whether a candidate $d \neq c$ can tie or beat c on level i via adding at most ℓ voters from V' . Since the first majority stage is slightly different from the subsequent majority stages, we describe both cases separately.

Majority stage 1 For each candidate $d \in C - \{c\}$, check whether d can tie or beat c on the first level via adding at most ℓ voters from V' . To this end, find a list $V'_d \subseteq V'$ of largest cardinality such that $\|V'_d\| \leq \ell$ and all voters in V'_d approve of d on the first level. Check whether

$$\text{score}_{(C, V \cup V'_d)}^1(d) \geq \text{score}_{(C, V \cup V'_d)}^1(c). \quad (1)$$

If (1) fails to hold, this d is hopeless, so go to the next candidate (or to the next stage if all other candidates have already been checked in this stage). If (1) holds, check whether d has a strict majority in $(C, V \cup V'_d)$ on the first level, and if so, output $V'' = V'_d$ and halt. Otherwise, this d is hopeless, so go to the next candidate (or stage).

Majority stage i , $1 < i \leq k$ This stage is entered only if it was not possible to find a successful control action in majority stages $1, \dots, i-1$. For each candidate $d \in C - \{c\}$, check whether d can tie with or beat c up to the i th level via adding at most ℓ voters from V' . To this end, find a list $V'_d \subseteq V'$ of largest cardinality such that $\|V'_d\| \leq \ell$ and all voters in V'_d approve of d up to the i th level but disapprove of c up to the i th level. Check whether

$$\text{score}_{(C, V \cup V'_d)}^i(d) \geq \text{score}_{(C, V \cup V'_d)}^i(c). \quad (2)$$

If (2) fails to hold, this d is hopeless, so go to the next candidate (or to the next stage if all other candidates have already been checked in this stage). If (2) holds, check whether d has a strict majority in $(C, V \cup V'_d)$ on the i th level, and if so, check whether

$$\text{score}_{(C, V \cup V'_d)}^{i-1}(c) \geq \text{maj}(V \cup V'_d). \quad (3)$$

If (3) fails to hold, output $V'' = V'_d$ and halt. Otherwise (i.e., if (3) holds), though d might dethrone c by adding V'_d on the i th level, it was not quick enough: c has already won earlier. In that case, find a largest list $V'_{cd} \subseteq V'$ such that (i) $\|V'_d \cup V'_{cd}\| \leq \ell$, (ii) all voters in V'_{cd} approve of both c and d up to the i th level, and (iii) the voters in V'_{cd} are chosen such that c is approved of exactly on level i . Now, check whether

$$\text{score}_{(C, V \cup V'_d \cup V'_{cd})}^{i-1}(c) \geq \text{maj}(V \cup V'_d \cup V'_{cd}). \quad (4)$$

If (4) fails to hold, check whether $\|V'_{cd}\| \geq 2 \text{def}_{(C, V \cup V'_d)}^i(d)$. If so (note that d has now a strict majority on level i), output $V'' = V'_d \cup V'_{cd}$ and halt. Note that, by choice of V'_{cd} , (2) implies that

$$\text{score}_{(C, V \cup V'_d \cup V'_{cd})}^i(d) \geq \text{score}_{(C, V \cup V'_d \cup V'_{cd})}^i(c).$$

Thus, in $(C, V \cup V'_d \cup V'_{cd})$, d ties with or beats c and has a strict majority on the i th level (and now, we are sure that d was not too late). Otherwise (i.e., if $\|V'_{cd}\| < 2 \text{def}_{(C, V \cup V'_d)}^i(d)$), or if (4) holds, though d might dethrone c by adding $V'_d \cup V'_{cd}$ on the i th level, it was still not quick enough: c has already won earlier. In that case, find a largest list $V'_\emptyset \subseteq V'$ such that (i) $\|V'_d \cup V'_{cd} \cup V'_\emptyset\| \leq \ell$ and (ii) all voters in V'_\emptyset disapprove of both c and d up to the i th level. Check whether

$$\text{score}_{(C, V \cup V'_d \cup V'_{cd} \cup V'_\emptyset)}^{i-1}(c) \geq \text{maj}(V \cup V'_d \cup V'_{cd} \cup V'_\emptyset). \quad (5)$$

If (5) holds, then d is hopeless, so go to the next candidate (or to the next stage if all other candidates have already been checked in this stage). Otherwise (i.e., if (5) fails to hold), check whether

$$\begin{aligned} \|V'_\emptyset\| &\leq 2 \text{surplus}_{(C, V \cup V'_d \cup V'_{cd})}^i(d) \quad (\text{if } \|V \cup V'_d \cup V'_{cd}\| \text{ is odd}) \quad \text{or} \\ \|V'_\emptyset\| &< 1 + 2 \text{surplus}_{(C, V \cup V'_d \cup V'_{cd})}^i(d) \quad (\text{if } \|V \cup V'_d \cup V'_{cd}\| \text{ is even}). \end{aligned}$$

If so (note that d has a strict majority on level i), output $V'' = V'_d \cup V'_{cd} \cup V'_\emptyset$ and halt. Note that, by choice of V'_\emptyset , (2) again implies that $\text{score}_{(C, V \cup V'_d \cup V'_{cd} \cup V'_\emptyset)}^i(d) \geq \text{score}_{(C, V \cup V'_d \cup V'_{cd} \cup V'_\emptyset)}^i(c)$. Thus, in $(C, V \cup V'_d \cup V'_{cd} \cup V'_\emptyset)$, d ties with or beats c and has a strict majority on the i th level (and d was not too late). Otherwise (i.e., if $\|V'_\emptyset\| > 2 \text{surplus}_{(C, V \cup V'_d \cup V'_{cd})}^i(d)$ or $\|V'_\emptyset\| > 1 + 2 \text{surplus}_{(C, V \cup V'_d \cup V'_{cd})}^i(d)$ if $\|V \cup V'_d \cup V'_{cd}\|$ is odd or even, respectively), this d is hopeless, so go to the next candidate (or stage).

Approval stage This stage is entered only if it was not possible to find a successful control action in majority stages $1, 2, \dots, k$. First, check if

$$\text{score}_{(C, V)}(c) < \left\lfloor \frac{\|V\| + \ell}{2} \right\rfloor + 1. \quad (6)$$

If (6) fails to hold, output “control impossible” and halt, since we have found no candidate in the majority stages who could tie or beat c and would have a strict majority when adding at most ℓ voters from V' , so adding any choice of at most ℓ voters from V' would still leave c with a strict majority. If (6) holds, looping over all candidates $d \in C - \{c\}$, check whether there are $\text{score}_{(C, V)}(c) - \text{score}_{(C, V)}(d) \leq \ell$ voters in V' who approve of d and disapprove of c . If this is not the case, move

on to the next candidate, since d could never catch up on c via adding at most ℓ voters from V' . If it is the case for some $d \in C - \{c\}$, however, add this list of voters (call it V'_d) and check whether

$$\text{score}_{(C, V \cup V'_d)}(c) < \text{maj}(V \cup V'_d). \tag{7}$$

If (7) holds, output $V'' = V'_d$ and halt. Otherwise (i.e., if (7) fails to hold), check whether

$$\begin{aligned} \ell - \|V'_d\| &\geq \|V'_\emptyset\| \\ &\geq 2 \left(\text{score}_{(C, V \cup V'_d)}(c) - \frac{\|V \cup V'_d\|}{2} \right), \end{aligned} \tag{8}$$

where V'_\emptyset consists of those voters in V' who disapprove of both candidates, c and d . If (8) does not hold, move on to the next candidate, since after adding these voters c would still have a strict majority. Otherwise (i.e., if (8) holds), add exactly $2(\text{score}_{(C, V \cup V'_d)}(c) - \|V \cup V'_d\|/2)$ voters from V'_\emptyset (denoted by $V'_{\emptyset,+}$). Output $V'' = V'_d \cup V'_{\emptyset,+}$ and halt.

If we have entered the approval stage (because we were not successful in any of the majority stages), but couldn't find any candidate here who was able to dethrone c by adding at most ℓ voters from V' , we output “control impossible” and halt.

The correctness of the algorithm can be seen as follows. If candidate c is not the unique winner of the election, the algorithm correctly outputs \emptyset and terminates. Let $d \in C - \{c\}$ be a candidate who ties with or beats c on level i by adding at most ℓ votes, and suppose there is no other candidate who can tie with or beat c by adding at most ℓ voters on a level prior to i . We now argue that our algorithm will either find this candidate (and will also output the set of votes to add) or a different candidate who can tie with or beat c on level i after adding at most ℓ voters.

It is easy to see that the algorithm only stops in a majority stage if the following three necessary conditions are satisfied:

1. d 's level i score is at least as high as c 's level i score,
2. d has a strict majority on level i , and
3. c has no strict majority on level $i - 1$.

If the algorithm stops before checking candidate d in majority stage i , then it found a different candidate who could tie with or beat candidate c on level i . Otherwise, when checking candidate d on level i , (2) and the fact that we do not add voters who approve of c and disapprove of d until level i guarantees that d ties with or beats c by score on level i . Furthermore, by adding the maximal sized voter set, in which each voter approves of d until level i , the algorithm provides a strict majority (which exists according to our starting assumption) on level i for candidate d . Because of the algorithm's “surplus-check,” this strict majority will not be destroyed even after adding (in some cases) some voters who do not approve of d . Finally, depending on how many voter groups the algorithm adds, (3), (4), or (5) ensure that c has not a strict majority on level $i - 1$.

It remains to deal with the issue that there is no candidate who can dethrone c on any of the majority stages. Since there is no candidate satisfying the three necessary conditions, the algorithm would not stop until reaching the approval stage. Clearly, the algorithm only outputs a successful addition of voters if c has no strict majority on any level and there exists a candidate whose approval score is at least as high as c 's. Otherwise, the algorithm gives the correct answer that control is impossible and halts. Crucially, note that the algorithm proceeds in the “safest way possible”: If there is any successful control action then our algorithm finds some successful control action.

The algorithm runs in polynomial time as it proceeds in at most $k + 1 \leq m + 1$ stages, where in each stage certain conditions are checked for up to $m - 1$ candidates. For checking the conditions we have to go through the list V' and compute level scores over at most $n + n'$ voters, each a constant number of times. Thus, the algorithm's worst-case runtime is in $\mathcal{O}(m^2(n' + p(n + n')))$. (Recall that we didn't optimize it in terms of its runtime; rather, we described it in a way to make it easier to check its correctness.)

Note that by changing the “greater than or equal to” conditions to “strictly greater than” conditions every time the algorithm compares the score of d and c , the same algorithm works in the co-winner model, too. \square

Corollary 3.14. *Bucklin voting is vulnerable to destructive control by adding voters in both winner models.*

We now turn to the case of deleting voters and we start with the constructive control scenario.

Theorem 3.15. *Bucklin voting is resistant to constructive control by deleting voters, and is parameterizedly resistant when this control problem is parameterized by the number of voters deleted. Each result holds in both winner models.*

Proof. In light of Remark 2.4, we again will give only one reduction that proves both claims at the same time. To prove $W[2]$ -hardness in the unique-winner model, we again provide a reduction from k -DOMINATING SET. Let (G, k) with $G = (B, E)$ be a given instance of this problem as described at the start of this section. Define the election (C, V) , where $C = \{c, w\} \cup B \cup X \cup Y \cup Z$ is the set of candidates, w is the distinguished candidate, and X, Y , and Z are sets of padding candidates (recall Footnote 12).

Table 4
Level i scores in (C, V) for $i \in \{1, n+1\}$ and the candidates in $B \cup \{c, w\}$.

	c	w	$b_j \in B$
$score^1$	$k - 1$	0	$\leq n$
$score^{n+1}$	$n + k - 1$	n	n

Padding candidates in X: X is a set of $\sum_{i=1}^n \|B - N[b_i]\|$ candidates such that for each i , $1 \leq i \leq n$, we can find a subset $X_i \subseteq X$ with $n - \|N[b_i]\|$ elements such that $X_r \cap X_s = \emptyset$ for all $r, s \in \{1, \dots, n\}$ with $r \neq s$. These subsets ensure that c is always placed among the top $(n + 1)$ positions in the first voter group of V below.

Padding candidates in Y: Y is a set of $\sum_{i=1}^n \|N[b_i]\|$ candidates such that for each i , $1 \leq i \leq n$, we can find a subset $Y_i \subseteq Y$ with $\|N[b_i]\|$ elements such that $Y_r \cap Y_s = \emptyset$ for all $r, s \in \{1, \dots, n\}$ with $r \neq s$. These subsets ensure that w is always placed at the $(n + 1)$ st position in the second voter group of V below.

Padding candidates in Z: Z is a set of $(k - 1)(n + 1)$ candidates such that for each j , $1 \leq j \leq k - 1$, we can find a subset $Z_j \subseteq Z$ with $n + 1$ elements such that $Z_r \cap Z_s = \emptyset$ for all $r, s \in \{1, \dots, k - 1\}$ with $r \neq s$. These subsets ensure that no other candidate besides c and the candidates in Z_j gain any points up to the $(n + 2)$ nd level in the third voter group of V below.

V is the following collection of $2n + k - 1$ voters:

#	For each ...	number of votes	ranking of candidates
1	$i \in \{1, \dots, n\}$	1	$N[b_i] \ c \ X_i \ ((B - N[b_i]) \cup (X - X_i) \cup Y \cup Z) \ w$
2	$i \in \{1, \dots, n\}$	1	$(B - N[b_i]) \ Y_i \ w \ (N[b_i] \cup X \cup (Y - Y_i) \cup Z \cup \{c\})$
3	$j \in \{1, \dots, k - 1\}$	1	$c \ Z_j \ (B \cup X \cup Y \cup (Z - Z_j)) \ w$

The relevant scores in (C, V) can be seen in Table 4 above.

It holds that $n + k - 1 > maj(V) > n$. Since candidate w reaches a strict majority only on the last level but c does so no later than on the $(n + 1)$ st level, w is not a unique Bucklin winner of this election.

We claim that G has a dominating set of size k if and only if w can be made the unique Bucklin winner by deleting at most k voters.

From left to right: Suppose G has a dominating set B' of size k . Delete the corresponding voters from the first voter group (i.e., each voter v_i for which $b_i \in B'$). Let V' denote the resulting set of voters and note that $\|V'\| = 2n - 1$. Now, in election (C, V') we have on level $n + 1$:

- $score_{(C, V')}^{n+1}(b_j) \leq n - 1$ for each $b_j \in B$ (from the first and second voter groups; no b_j can have a score of n on level $n + 1$, since B' is a dominating set in G , so $B = N[B']$, and all voters v_i corresponding to members b_i of B' have been deleted),
- $score_{(C, V')}^{n+1}(c) = (n - k) + (k - 1) = n - 1$ (from the first and third voter groups),
- $score_{(C, V')}^{n+1}(x_i) = 1$ for each $x_i \in X$ (from the first voter group),
- $score_{(C, V')}^{n+1}(y_i) = 1$ for each $y_i \in Y$ (from the second voter group),
- $score_{(C, V')}^{n+1}(z_i) = 1$ for each $z_i \in Z$ (from the third voter group), and
- $score_{(C, V')}^{n+1}(w) = n$ (from the second voter group).

That is, only candidate w reaches a strict majority on level $n + 1$ in (C, V') , so w is the unique Bucklin winner of this election.

From right to left: Suppose w can be made the unique Bucklin winner by deleting at most k voters. Let V' be the set of remaining voters. Observe that deleting less than k voters would make it impossible for candidate w to be a unique Bucklin winner of the election. Indeed, if less than k voters are deleted from V , the strict majority threshold for the set V' of remaining voters would exceed n . However, since w is ranked last place in all votes except the n votes from the second voter group, w would reach a strict majority no earlier than on the last level and thus would not be a unique Bucklin winner of this election. Clearly, w has to win election (C, V') on level $n + 1$. Since $score_{(C, V')}^{n+1}(b_i) = n = score_{(C, V')}^{n+1}(w)$ for all i with $1 \leq i \leq n$, by deleting these k votes from V each b_i has to lose at least one point on the first $n + 1$ levels. Obviously, no voters from the second voter group can be deleted, for otherwise candidate w would not reach the strict majority threshold on level $n + 1$. Similarly, deleting voters from the third voter group does not make any $b_i \in B$ lose any points up to level $n + 1$. So at least part of the deleted voters have to be from the first voter group, let us say we delete $k' \leq k$. Since every candidate $b_i \in B$ has to lose at least one point up to level $n + 1$, the k' deleted voters in $V - V'$ correspond to a dominating set in G . If $k' < k$, we can delete voters arbitrarily from the first and/or third voter group until the total allowed number of k deleted voters is reached (that is needed to ensure the right majority threshold in the new election).

Note that this polynomial-time reduction is parameterized, as the given parameter k of k -DOMINATING SET is the same parameter k that bounds the number of voters that may be deleted in this control problem.

To show (parameterized) resistance also in the co-winner model, the voter list has to be slightly adapted: One voter of group 3 has to be deleted (thus, $j \in \{1, \dots, k - 2\}$) and one voter (the only one in a new, fourth group) with preference $Bc(X \cup Y \cup Z)$ has to be added. □

Corollary 3.16. *Fallback voting is resistant to constructive control by deleting voters, and is parameterizedly resistant when this control problem is parameterized by the number of voters deleted. Each result holds in both winner models.*

The following theorem is stated without proof, since the algorithm for destructive control by adding voters presented in the proof of Theorem 3.13 can easily be adapted to the deleting-voters case (and listing all changes would be long and tedious).

Theorem 3.17. *Fallback voting is vulnerable to destructive control by adding voters, in both winner models.*

Corollary 3.18. *Bucklin voting is vulnerable to destructive control by adding voters, in both winner models.*

We now turn to control by partition of voters.

Theorem 3.19. *Bucklin voting is resistant to constructive control by partition of voters in both tie-handling models, TE and TP, each in both winner models.*

Proof. Susceptibility holds by Lemma 3.1. To show NP-hardness we reduce X3C to our control problems. Let (B, \mathcal{S}) be an X3C instance with $B = \{b_1, b_2, \dots, b_{3m}\}$, $m > 1$, and a collection $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i , $1 \leq i \leq n$. We define the election (C, V) , where $C = B \cup \{c, w, x\} \cup D \cup E \cup F \cup G$ is the set of candidates, w is the distinguished candidate, and D, E, F , and G are sets of padding candidates (recall Footnote 12).

Subsets B_1, B_2, \dots, B_n of B : These are n subsets of B (in general, not disjoint) that are defined such that each candidate in B gains exactly n points in total up to level $3m$ from the first and the second voter group of V below. With $\ell_j = \|\{S_i \in \mathcal{S} \mid b_j \in S_i\}\|$ for each j , $1 \leq j \leq 3m$, these subsets are formally defined by $B_i = \{b_j \in B \mid i \leq n - \ell_j\}$ for each i , $1 \leq i \leq n$.

Padding candidates in D : D is a set of $3nm$ candidates such that for each i , $1 \leq i \leq n$, we can find a subset $D_i \subseteq D$ with $3m - \|B_i\|$ elements such that $D_r \cap D_s = \emptyset$ for all $r, s \in \{1, \dots, n\}$ with $r \neq s$. These subsets ensure that w is always placed at position $3m + 1$ in the second voter group of V below.

Padding candidates in E : E is a set of $(3m - 1)(m + 1)$ candidates such that for each k , $1 \leq k \leq m + 1$, we can find a subset $E_k \subseteq E$ with $3m - 1$ elements such that $E_r \cap E_s = \emptyset$ for all $r, s \in \{1, \dots, m + 1\}$ with $r \neq s$. These subsets ensure that no other candidate besides c and x gains more than one point up to the $(3m + 1)$ st level in the third voter group of V below.

Padding candidates in F : F is a set of $(3m + 1)(m - 1)$ candidates such that for each l , $1 \leq l \leq m - 1$, we can find a subset $F_l \subseteq F$ with $3m + 1$ elements such that $F_r \cap F_s = \emptyset$ for all $r, s \in \{1, \dots, m - 1\}$ with $r \neq s$. These subsets ensure that c does not gain any points up to level $3m + 1$ in the fourth voter group of V below.

Padding candidates in G : G is a set of $n(3m - 3)$ candidates such that for each i , $1 \leq i \leq n$, we can find a subset $G_i \subseteq G$ with $3m - 3$ elements such that $G_r \cap G_s = \emptyset$ for all $r, s \in \{1, \dots, n\}$ with $r \neq s$. These subsets ensure that no other candidate besides c and those in S_i gains more than one point up to level $3m + 1$ in the first voter group of V below.

Let V consist of the following $2n + 2m$ voters:

#	For each ...	number of votes	ranking of candidates
1	$i \in \{1, \dots, n\}$	1	$c \ S_i \ G_i \ (G - G_i) \ F \ D \ E \ (B - S_i) \ w \ x$
2	$i \in \{1, \dots, n\}$	1	$B_i \ D_i \ w \ G \ E \ (D - D_i) \ F \ (B - B_i) \ c \ x$
3	$k \in \{1, \dots, m + 1\}$	1	$x \ c \ E_k \ F \ (E - E_k) \ G \ D \ B \ w$
4	$l \in \{1, \dots, m - 1\}$	1	$F_l \ c \ (F - F_l) \ G \ D \ E \ B \ w \ x$

The strict majority threshold is reached with $n + m + 1$ points and Table 5 shows the scores on the relevant levels in election (C, V) .

Clearly, candidate c is the unique level 2 BV winner in election (C, V) with a level 2 score of $n + m + 1$.

We claim that \mathcal{S} has an exact cover \mathcal{S}' for B if and only if w can be made the unique BV winner of the resulting election by partition of voters (regardless of the tie-handling model used).

Table 5
Level i scores in (C, V) for $i \in \{1, 2, 3m, 3m + 1\}$ and the candidates in $B \cup \{c, w, x\}$.

	c	b_j	w	x
$score^1$	n	$\leq n$	0	$m + 1$
$score^2$	$n + m + 1$	$\leq n$	0	$m + 1$
$score^{3m}$	$n + m + 1$	n	0	$m + 1$
$score^{3m+1}$	$n + m + 1$	n	n	$m + 1$

Table 6
Level i scores in (C, V_1) and (C, V_2) for $i \in \{1, 2, 3m, 3m + 1\}$ and the candidates in $B \cup \{c, w, x\}$.

	(C, V_1)			(C, V_2)		
	c	b_j	x	c	b_j	w
$score^1$	m	0	$m + 1$	$n - m$	$\leq n - 1$	0
$score^2$	$2m + 1$	≤ 1	$m + 1$	$n - m$	$\leq n - 1$	0
$score^{3m}$	$2m + 1$	1	$m + 1$	$n - m$	$n - 1$	0
$score^{3m+1}$	$2m + 1$	1	$m + 1$	$n - m$	$n - 1$	n

Table 7
Level i scores of w and x in the final election $(\{w, x\}, V)$ for $i \in \{1, 2\}$.

	x	w
$score^1$	$m + 1$	$2n + m - 1$
$score^2$	$2n + 2m$	$2n + 2m$

From left to right: Suppose \mathcal{S} has an exact cover \mathcal{S}' for B . Partition V as follows. Let V_1 consist of:

- the m voters of the first group that correspond to the exact cover (i.e., those m voters of the form $c S_i G_i (G - G_i) F D E (B - S_i) w x$ for which $S_i \in \mathcal{S}'$) and
- the $m + 1$ voters of the third group (i.e., all voters of the form $x c E_k F (E - E_k) G D B w$).

Let $V_2 = V - V_1$. Table 6 shows the relevant scores in the two subelections.

In subelection (C, V_1) , candidate x is the unique level 1 BV winner. In subelection (C, V_2) , candidate w is the first candidate who has a strict majority and moves on to the final round of the election. Thus there are w and x in the final run-off, which w wins with a strict majority on the first level as can be seen in Table 7 presenting the scores of x and w in the final election.

Since both subelections, (C, V_1) and (C, V_2) , have unique BV winners, candidate w can be made the unique BV winner by partition of voters, regardless of the tie-handling model used.

From right to left: Suppose that w can be made the unique BV winner by exerting control by partition of voters (for concreteness, say in TP). Let (V_1, V_2) be such a successful partition. Since w wins the resulting two-stage election, w has to win at least one of the subelections (say, w wins (C, V_2)). If candidate c participates in the final round, he or she wins the election with a strict majority no later than on the second level, no matter which other candidates move forward to the final election. That means that in both subelections, (C, V_1) and (C, V_2) , c must not be a BV winner. Only in the second voter group candidate w (who has to be a BV winner in election (C, V_2)) gets points earlier than on the second-to-last level. So w has to be a level $3m + 1$ BV winner in election (C, V_2) via votes from the second voter group in V_2 . As c scores already on the first two levels in voter groups 1 and 3, only x and the candidates in B can prevent c from winning in (C, V_1) . However, since voters from the second voter group have to be in V_2 (as stated above), in subelection (C, V_1) only candidate x can prevent c from moving forward to the final round. Since x is always placed behind c in all votes except those votes from the third voter group, x has to be a level 1 BV winner in election (C, V_1) . In (C, V_2) candidate w gains all the points on exactly the $(3m + 1)$ st level, whereas the other candidates scoring more than one point up to this level receive their points on either earlier or later levels, so no candidate can tie with w on the $(3m + 1)$ st level and w is the unique level $3m + 1$ BV winner in election (C, V_2) . As both subelections, (C, V_1) and (C, V_2) , have unique BV winners other than c , the construction works in model TE as well.

It remains to show that \mathcal{S} has an exact cover \mathcal{S}' for B . Since w has to win (C, V_2) with the votes from the second voter group, not all voters from the first voter group can be in V_2 (otherwise c would have n points already on the first level). On the other hand, there can be at most m voters from the first voter group in V_1 because otherwise x would not be a level 1 BV winner in election (C, V_1) . To ensure that no candidate contained in B has the same score as w , namely n points, and gets these points on an earlier level than w in (C, V_2) , there have to be exactly m voters from the first group in V_1 and these voters correspond to an exact cover for B .

Without further adaptations, this reductions also covers the co-winner case. □

Table 8
Level i scores for $i \in \{1, 2, m + 2\}$ in the election (C, V) from [Construction 3.23](#).

	c	w	$b_j \in B$	$d_p \in D$	$e_r \in E$
score ¹	$n(k + 1) + 2m + mk$	$n(k + 1) + 1$	$k - 1$	≤ 1	1
score ²	$n(k + 1) + 2m + mk + 1$	$n(k + 1) + mk + k$	$\leq k + n(k + 1)$	1	1
score ³	$\leq 2n(k + 1) + 2m + mk + 1$	$n(k + 1) + 2m + mk + k + 1$	$\leq k + n(k + 1)$	1	1
score ^{$m+2$}	$2n(k + 1) + 2m + mk + 1$	$n(k + 1) + 2m + mk + k + 1$	$\leq k + n(k + 1)$	1	1

Corollary 3.20. *Fallback voting is resistant to constructive control by partition of voters in model TE and model TP, each in both winner models.*

We now turn to destructive control by partition of voters in model TE where we show resistance for Bucklin and fallback voting via a reduction from the dominating set problem that has been defined in Section 2.3. The proof of [Theorem 3.21](#) is deferred to [Appendix A](#).

Theorem 3.21. *Bucklin voting is resistant to destructive control by partition of voters in model TE for both winner models.*

Resistance (and parameterized resistance) for fallback voting to this control type now follows immediately.

Corollary 3.22. *Fallback voting is resistant to destructive control by partition of voters in model TE for both winner models.*

The following construction will be used to handle the case of destructive control by partition of voters in model TP (see [Theorem 3.25](#) below) for fallback voting. [Construction 3.23](#) starts from an instance of the NP-complete problem RHS defined in Section 2.3. This construction does not work for Bucklin voting, and we leave the complexity of this case (Bucklin-DCPV-TP) open.

Construction 3.23. *Let (B, S, k) be a given instance of RHS, where $B = \{b_1, b_2, \dots, b_m\}$ is a set, $S = \{S_1, S_2, \dots, S_n\}$ is a collection of nonempty subsets $S_i \subseteq B$ such that $n > m$, and k is an integer with $1 < k < m$. Define the election (C, V) , where $C = B \cup D \cup E \cup \{c, w\}$ is the candidate set with $D = \{d_1, \dots, d_{2(m+1)}\}$ and $E = \{e_1, \dots, e_{2(m-1)}\}$. Note that the candidates contained in D and E are padding candidates (recall [Footnote 12](#)). The candidates in D ensure that w is always placed at the third position in the votes of the fourth voter group of V below. The collection of voters V consists of the following $2n(k + 1) + 4m + 2mk$ voters:*

#	For each ...	number of voters	ranking of approved candidates
1	$i \in \{1, \dots, n\}$	$k + 1$	$w \ S_i \ c$
2	$j \in \{1, \dots, m\}$	1	$c \ b_j \ w$
3	$j \in \{1, \dots, m\}$	$k - 1$	b_j
4	$p \in \{1, \dots, m + 1\}$	1	$d_{2(p-1)+1} \ d_{2p} \ w$
5	$r \in \{1, \dots, 2(m - 1)\}$	1	e_r
6		$n(k + 1) + m - k + 1$	c
7		$mk + k - 1$	$c \ w$
8		1	$w \ c$

The strict majority threshold for V is $maj(V) = n(k + 1) + 2m + mk + 1$. In election (C, V) , only the two candidates c and w reach a strict majority, w on the third level and c on the second level (see [Table 8](#)). Thus c is the unique level 2 FV winner of election (C, V) .

[Lemma 3.24](#) will be used in the proof of [Theorem 3.25](#).

Lemma 3.24. *In the election (C, V) from [Construction 3.23](#), for every partition of V into V_1 and V_2 , candidate c is an FV winner of election (C, V_1) or (C, V_2) .*

Proof. For a contradiction, suppose that in both subelections, (C, V_1) and (C, V_2) , candidate c is not an FV winner. Since $score^1_{(C, V)}(c) = \|V\|/2$, the two subelections must satisfy that both $\|V_1\|$ and $\|V_2\|$ are even numbers, and $score^1_{(C, V_1)}(c) = \|V_1\|/2$ and $score^1_{(C, V_2)}(c) = \|V_2\|/2$. Otherwise, c would have a strict majority already on the first level in one of the subelections and would win that subelection. For each $i \in \{1, 2\}$, c already on the first level has only one point less than the strict majority threshold $maj(V_i)$ in subelection (C, V_i) , and c will get a strict majority in (C, V_i) no later than the $(m + 2)$ nd level. Thus, for both $i = 1$ and $i = 2$, there must be candidates whose level $m + 2$ scores in (C, V_i) are higher than the level $m + 2$ score of c in (C, V_i) . [Table 8](#) shows the level $m + 2$ scores of all candidates in (C, V) . Only w and some $b_j \in B$ have a chance to beat c on that level in (C, V_i) , $i \in \{1, 2\}$.

Table 9
Level i scores in (C, V_1) for $i \in \{1, 2, 3\}$ and the candidates in $B \cup \{c, w\}$.

	c	w	$b_j \in B'$	$b_j \notin B'$
score ¹	k	0	$k - 1$	0
score ²	k	0	k	0
score ³	k	k	k	0

Table 10
Level i scores in (C, V_2) for $i \in \{1, 2, 3\}$ and the candidates in $B \cup \{c, w\}$.

	c	w	$b_j \notin B'$	$b_j \in B'$
score ¹	$n(k + 1) + 2m - k + mk$	$n(k + 1) + 1$	$k - 1$	0
score ²	$n(k + 1) + 2m - k + mk + 1$	$n(k + 1) + mk + k$	$\leq k + n(k + 1)$	$\leq n(k + 1)$
score ³	$\geq n(k + 1) + 2m - k + mk + 1$	$n(k + 1) + mk + 2m + 1$	$\leq k + n(k + 1)$	$\leq n(k + 1)$

Suppose that c is defeated in both subelections by two distinct candidates from B (say, b_x defeats c in (C, V_1) and b_y defeats c in (C, V_2)). Thus the following must hold:¹⁴

$$\begin{aligned} \text{score}_{(C, V_1)}^{m+2}(b_x) + \text{score}_{(C, V_2)}^{m+2}(b_y) &\geq \text{score}_{(C, V)}^{m+2}(c) + 2 \\ 2n(k + 1) + 2k - n(k + 1) &\geq 2n(k + 1) + mk + 2m + 3 \\ 2k &\geq n(k + 1) + mk + 2m + 3, \end{aligned}$$

which by our basic assumption $m > k > 1$ implies the following contradiction:

$$0 \geq n(k + 1) + (m - 2)k + 2m + 3 > n(k + 1) + (k - 2)k + 2k + 3 = n(k + 1) + k^2 + 3 > 0.$$

Thus the only possibility for c to not win any of the two subelections is that c is defeated in one subelection, say (C, V_1) , by a candidate from B , say b_x , and in the other subelection, (C, V_2) , by candidate w . Then it must hold that (again, see Footnote 14):

$$\begin{aligned} \text{score}_{(C, V_1)}^{m+2}(b_x) + \text{score}_{(C, V_2)}^{m+2}(w) &\geq \text{score}_{(C, V)}^{m+2}(c) + 2 \\ 2n(k + 1) + 2k + 2m + mk + 1 - n(k + 1) - 1 &\geq 2n(k + 1) + mk + 2m + 3 \\ 2k &\geq n(k + 1) + 3. \end{aligned}$$

Since $n > 1$, this cannot hold, so c must be an FV winner in one of the two subelections. \square

Theorem 3.25. *Fallback voting is resistant to destructive control by partition of voters in model TP, in both winner models.*

Proof. Susceptibility holds by Corollary 3.2. To prove NP-hardness in the unique-winner model, we reduce RHS to our control problem. Consider the election (C, V) constructed according to Construction 3.23 from a given RHS instance (B, \mathcal{S}, k) , where $B = \{b_1, \dots, b_m\}$ is a set, $\mathcal{S} = \{S_1, \dots, S_n\}$ is a collection of nonempty subsets $S_i \subseteq B$, and k is an integer with $1 < k < m < n$.

We claim that \mathcal{S} has a hitting set $B' \subseteq B$ of size k if and only if c can be prevented from being a unique FV winner by partition of voters in model TP.

From left to right: Suppose, $B' \subseteq B$ is a hitting set of size k for \mathcal{S} . Partition V into V_1 and V_2 the following way. Let V_1 consist of those voters of the second group where $b_j \in B'$ and of those voters of the third group where $b_j \in B'$. Let $V_2 = V - V_1$. In (C, V_1) , no candidate reaches a strict majority (see Table 9), where $\text{maj}(V_1) = \lfloor k^2/2 \rfloor + 1$, and candidates c , w , and each $b_j \in B'$ win the election with an approval score of k .

The level i scores in election (C, V_2) for $i \in \{1, 2, 3\}$ and the candidates in $B \cup \{c, w\}$ are shown in Table 10. Since in (C, V_2) no candidate from B wins, the candidates participating in the final round are $B' \cup \{c, w\}$. The scores in the final election $(B' \cup \{c, w\}, V)$ can be seen in Table 11. Since candidates c and w with the same level 2 scores are both level 2 FV winners, candidate c has been prevented from being a unique FV winner by partition of voters in model TP.

From right to left: Suppose candidate c can be prevented from being a unique FV winner by partition of voters in model TP. From Lemma 3.24 it follows that candidate c participates in the final round. Since c has a strict majority of approvals, c has to be tied with or lose against another candidate by a strict majority at some level. Only candidate w has a strict majority of approvals, so w has to tie or beat c at some level in the final round. Because of the low scores of the candidates in D and E we may assume that only candidates from B are participating in the final round besides c and w .

¹⁴ For the left-hand sides of the inequalities, note that each vote occurs in only one of the two subelections. To avoid double-counting those votes that give points to both candidates, we first sum up the overall number of points each candidate scores and then subtract the double-counted points.

Table 11Level i scores in the final-stage election $(B' \cup \{c, w\}, V)$ for $i \in \{1, 2\}$.

	c	w	$b_j \in B'$
score ¹	$n(k+1) + 2m + mk$	$n(k+1) + m + 2$	$k - 1$
score ²	$n(k+1) + 2m + mk + 1$	$n(k+1) + 2m + mk + 1$	$\leq k + n(k+1)$

Let $B' \subseteq B$ be the set of candidates who also participate in the final round. Let ℓ be the number of sets in S not hit by B' . As w cannot reach a strict majority of approvals on the first level, we consider the level 2 scores of c and w :

$$\text{score}_{(B' \cup \{c, w\}, V)}^2(c) = n(k+1) + 2m + mk + 1 + \ell(k+1),$$

$$\text{score}_{(B' \cup \{c, w\}, V)}^2(w) = n(k+1) + 2m + mk + k - \|B'\| + 1.$$

Since c has a strict majority already on the second level, w must tie or beat c on this level, so the following must hold:

$$\text{score}_{(B' \cup \{c, w\}, V)}^2(c) - \text{score}_{(B' \cup \{c, w\}, V)}^2(w) \leq 0$$

$$n(k+1) + 2m + mk + 1 + \ell(k+1) - n(k+1) - 2m - mk - k + \|B'\| - 1 \leq 0$$

$$\|B'\| - k + \ell(k+1) \leq 0.$$

This is possible only if $\ell = 0$ (i.e., all sets in S are hit by B'), which implies $\|B'\| \leq k$. Thus S has a hitting set of size at most k .

For this result, the adaption to the co-winner case is slightly more involved: First, one voter from group 6 has to be deleted. To compensate for this voter, a new voter group 9 has to be added, consisting of one voter with the preference $c f w$, where f is a new dummy candidate who is not approved by any voter except for one. This ensures that the remaining argumentation can be adapted straightforwardly while w gains one point on the second level in the final election and thus prevents c from being a winner. \square

4. Conclusions and open questions

We have shown that, among natural election systems with a polynomial-time winner problem, fallback voting displays the broadest control resistance currently known to hold. We have also shown that Bucklin voting behaves almost as well (possibly even as well) as fallback voting in terms of control resistance. In particular, both voting systems are resistant to all standard types of candidate control and to all standard types of constructive control, and they are vulnerable to only two types of destructive voter control. One case remains open: destructive control by partition of voters in the tie-handling model “ties promote” for Bucklin voting, so it might turn out that Bucklin has one more vulnerability. For comparison, recall from Table 1 that approval voting is vulnerable to destructive control by partition of voters both in model TE and in model TP and that SP-AV is vulnerable to this control type in model TE but resistant in model TP.

We also strengthened some of our resistance (i.e., NP-hardness) results for (unparameterized) control problems by showing that their parameterized variants for natural parameterizations are even W[2]-hard. It remains open whether or not these problems belong to W[2]. Another interesting task for future research along these lines is to find natural parameterizations for the other problems (including some problems modeling control by partition of either candidates or voters) and to study their parameterized complexity.

From a theoretical point of view, it would also be interesting and challenging to complement our (parameterized) resistance results by a typical-case analysis for control problems as has been done for manipulation (see, e.g., the work of Conitzer et al. [16], Procaccia and Rosenschein [60], Friedgut et al. [40], Isaksson et al. [50], Mossel and Racz [56], and Xia and Conitzer [64,65]). For experimental results on control in Bucklin and fallback voting, we refer to the companion paper [28] that is based on [62].

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Table 12
Level i scores in (C, V) for $i \in \{n-1, n, n+1\}$ and the candidates in $B \cup \{c, w\}$.

	$b_i \in B$	w	c
$score^{n-1}$	$\leq n$	1	0
$score^n$	$\leq n$	1	$n+1$
$score^{n+1}$	$\leq n$	$n+1$	$n+1$

Appendix A. Deferred proofs

A.1. Resistance to destructive control by deleting candidates

Proof of Theorem 3.4. In light of Remark 2.4 it is again enough to give just one reduction to prove both claims at the same time. Starting again with the unique-winner case, for the W[2]-hardness proof in the destructive case for Bucklin voting, let (G, k) with $G = (B, E)$ be a given instance of k -DOMINATING SET. Define the election (C, V) , where $C = \{c, w\} \cup B \cup M_1 \cup M_2 \cup M_3 \cup X \cup Y \cup Z$ is the candidate set, c is the distinguished candidate, and M_1, M_2, M_3, X, Y , and Z are sets of padding candidates (recall Footnote 12).

Padding candidates in M_1, M_2 , and M_3 : M_1, M_2 , and M_3 are three pairwise disjoint sets, where each is a set of k candidates that are positioned in the votes so as to ensure that no other candidate besides w and c can reach a strict majority up to level $n+k$.

Padding candidates in X : X is a set of $n^2 - \sum_{i=1}^n \|N[b_i]\|$ candidates such that for each $i, 1 \leq i \leq n$, we can find a subset $X_i \subseteq X$ with $n - \|N[b_i]\|$ elements such that $X_r \cap X_s = \emptyset$ for all $r, s \in \{1, \dots, n\}$ with $r \neq s$. These subsets ensure that w is always placed at the $(n+1)$ st position in the first voter group of V below.

Padding candidates in Y : Y is a set of $n-1$ padding candidates ensuring that c is at position n in the votes of the second voter group of V below.

Padding candidates in Z : Z is a set of $n-2$ padding candidates ensuring that w is at position $n-1$ and c is at position n in the vote of the third voter group of V below.

V is the following collection of $2n+1$ voters, so we have a strict majority threshold of $n+1$:

#	For each ...	# of votes	ranking of candidates
1	$i \in \{1, \dots, n\}$	1	$N[b_i] X_i w M_1 ((B - N[b_i]) \cup M_2 \cup M_3 \cup (X - X_i) \cup Y \cup Z) c$
2		n	$Y c M_2 (B \cup M_1 \cup M_3 \cup X \cup Z \cup \{w\})$
3		1	$Z w c M_3 (B \cup M_1 \cup M_2 \cup X \cup Y)$

Table 12 gives an overview of the scores on the relevant levels of the relevant candidates in election (C, V) . Note that candidate c is the unique level n Bucklin winner of election (C, V) , since c is the first candidate reaching a strict majority of votes (namely, $n+1$ points on level n , as indicated—here and in later score tables as well—by a boldfaced entry).

We claim that G has a dominating set of size k if and only if c can be prevented from being a unique Bucklin winner by deleting at most k candidates.

From left to right: Suppose G has a dominating set $B' \subseteq B$ of size k . Delete the corresponding candidates. Now candidate w moves at least one position to the left in each of the n votes in the first voter group. Since candidate c reaches a strict majority no earlier than on level n and $score_{(C-B', V)}^n(w) = n+1 = score_{(C-B', V)}^n(c)$, candidate c is no longer a unique Bucklin winner of the resulting election.

From right to left: Suppose c can be prevented from being a unique Bucklin winner of the election by deleting at most k candidates. Note that deleting one candidate from an election can move the strict majority level of another candidate at most one level to the left. Observe that only candidate w can prevent c from winning the election, since w is the only candidate other than c who reaches a strict majority of votes until level $n+k$. In election (C, V) , candidate w reaches this majority no earlier than on level $n+1$, and candidate c not before level n . Thus w can prevent c from being a unique winner only by scoring at least as many points as c no later than on level n . This is possible only if w is pushed at least one position to the left in all votes of the first voter group. By an argument analogous to that given in the constructive case for this control type (see the proof of Theorem 3.3), this implies that G has a dominating set of size k .

Note that this polynomial-time reduction is parameterized, as the given parameter k of k -DOMINATING SET is the same parameter k that bounds the number of candidates allowed to be deleted in the control problem.

To handle the co-winner model, one additional voter has to be added to V , with the preference $w(M_1 \cup M_2 \cup M_3 \cup X \cup Y \cup Z)(B \cup \{c\})$. This ensures that on level $n+1$ candidate w has one point more than c and can thus beat c strictly if and only if there is a dominating set of size at most k (which can be shown with an analogous argument to the one presented above). \square Theorem 3.4

Table 13
Level i scores of c , w , and x in (C, V) for $i \in \{1, 2, 3\}$.

	c	w	x
score ¹	$k + n$	0	k
score ²	$k + n$	1	k
score ³	$k + n + 1$	1	k

A.2. Resistance to destructive control by partition of voters in model TE

The following construction will be used in the proof of Theorem 3.21.

Construction A.1. Let $((B, A), k)$ be a given instance of DOMINATING SET with $B = \{b_1, b_2, \dots, b_n\}$ and $n \geq 1$. Define the election (C, V) with candidate set $C = B \cup D \cup E \cup F \cup H \cup \{c, u, v, w, x, y\}$, where c is the distinguished candidate, D, E, F, H , and $\{u, v\}$ are sets of padding candidates (recall Footnote 12), and y is a “partition-enforcing” candidate (see below).

Padding candidates in D : D is a set of $(k - 1)(n + 4)$ candidates such that for each $j, 1 \leq j \leq k - 1$, we can find a subset $D_j \subseteq D$ with $n + 4$ elements such that $D_r \cap D_s = \emptyset$ for all $r, s \in \{1, \dots, k - 1\}$ with $r \neq s$. These subsets ensure that no other candidate besides x gains more than one point up to level $n + 5$ in the third voter group of V below.

Padding candidates in E : E is a set of $2(k + n)$ candidates such that for each $l, 1 \leq l \leq k + n$, we can find a subset $E_l \subseteq E$ with two elements and it holds that $E_r \cap E_s = \emptyset$ for all $r, s \in \{1, \dots, k + n\}$ with $r \neq s$. These subsets ensure that x and y do not gain any points up to the fourth level in the fourth voter group of V below.

Padding candidates in F : F is a set of $3n$ candidates such that for each $i, 1 \leq i \leq n$, we can find a subset $F_i \subseteq F$ with three elements such that $F_r \cap F_s = \emptyset$ for all $r, s \in \{1, \dots, n\}$ with $r \neq s$. These subsets ensure that the candidates in B do not gain any points up to the fourth level in the first voter group of V below.

Padding candidates in H : H is a set of n^2 candidates such that for each $i, 1 \leq i \leq n$, we can find a subset $H_i \subseteq H$ with $\|N[b_i]\|$ elements such that $H_r \cap H_s = \emptyset$ for all $r, s \in \{1, \dots, n\}$ with $r \neq s$. These subsets ensure that w does not gain any points up to level $n + 5$ in the first voter group of V below.

Padding candidates u and v : These two candidates ensure that the other padding candidates are not among the top $n + 5$ positions in the second voter group of V below.

Partition-enforcing candidate y : This candidate ensures that the voter from the second voter group of V below has to be in the subelection candidate w wins to finally beat c in the final election.

V consists of the following $2k + 2n$ votes that can be arranged in four groups:

#	For each ...	number of votes	ranking of candidates
1	$i \in \{1, \dots, n\}$	1	$F_i \ (B - N[b_i]) \ H_i \ y \ w \dots$ $\dots (N[b_i] \cup D \cup E \cup (F - F_i) \cup (H - H_i)) \ u \ v \ c \ x$
2		1	$x \ w \ c \ B \ u \ v \ (D \cup E \cup F \cup H) \ y$
3	$j \in \{1, \dots, k - 1\}$	1	$x \ D_j \ (B \cup (D - D_j) \cup E \cup F \cup H) \ u \ v \ y \ w \ c$
4	$l \in \{1, \dots, k + n\}$	1	$c \ E_l \ x \ y \ (B \cup D \cup (E - E_l) \cup F \cup H) \ u \ v \ w$

Table 13 shows the scores of c, w , and x on the first three levels. None of the other candidates scores more than one point up to the third level. Note that c reaches a strict majority on this level and thus is the unique level 3 BV winner in this election.

Lemma A.2. In the election (C, V) from Construction A.1, for every partition of V into V_1 and V_2 , candidate c is the unique BV winner of at least one of the subelections, (C, V_1) and (C, V_2) .

Proof. For a contradiction, we assume that in both subelections, (C, V_1) and (C, V_2) , candidate c is not a unique BV winner. Table 13 shows that half of the voters in V place c already on the first level. Thus, for $i \in \{1, 2\}$, $\|V_i\|$ must be an even number and $score_{(C, V_i)}^1(c) = \|V_i\|/2$. Due to the voter in the second voter group, candidate c will get a strict majority on the third level in one of the subelections, let us say in (C, V_1) . So there has to be a candidate beating or tying with candidate c on the second or third level in (C, V_1) . The candidates in B, D, E, F , and H and the candidates u, v, w , and y do not score more than one point up to the third level. Thus only candidate x can possibly beat or tie with c on the second or third level in (C, V_1) . However, since x does not score more than k points in total until the fourth level, c is the unique level 3 BV winner in subelection (C, V_1) , a contradiction. It follows that c is a unique BV winner of at least one of the subelections. □

Using Lemma A.2, we are now ready to prove Theorem 3.21.

Table 14
Level $n + 5$ scores in (C, V_1) .

	c	w	x	y	$b_i \in B$
$score^{n+5}$	1	$k + 1$	k	k	$\leq k$

Proof of Theorem 3.21. Susceptibility follows from Lemma 3.1. We again provide a reduction from the NP-complete problem DOMINATING SET that has been defined in Section 2.3. Given a DOMINATING SET instance $((B, A), k)$, construct a Bucklin election (C, V) according to Construction A.1.

We claim that $G = (B, A)$ has a dominating set B' of size k if and only if candidate c can be prevented from being a unique BV winner by partition of voters in model TE.

From left to right: Let B' be a dominating set for G of size k . Partition V into V_1 and V_2 as follows. Let V_1 consist of the following $2k$ voters:

- The k voters of the first voter group corresponding to the dominating set, i.e., for those i with $b_i \in B'$, we have one voter of the form: $F_i (B - N[b_i]) H_i y w (N[b_i] \cup D \cup E \cup (F - F_i) \cup (H - H_i)) u v c x$,
- the one voter from the second group: $x w c B u v (D \cup E \cup F \cup H) y$, and
- the entire third voter group, i.e., for each j , $1 \leq j \leq k - 1$, there is one voter of the form: $x D_j (B \cup (D - D_j) \cup E \cup F \cup H) u v y w c$.

Let $V_2 = V - V_1$. Note that the strict majority threshold in V_1 is $maj(V_1) = k + 1$. Again, since the candidates in D , E , F , and H do not score more than one point up to level $n + 5$, their level $n + 5$ scores are not shown in Table 14. The level $n + 5$ scores of the remaining candidates are shown in this table. Note that w reaches a strict majority of $k + 1$ on this level (and no other candidate reaches a strict majority on this or an earlier level). Hence, w is the unique level $n + 5$ BV winner in subelection (C, V_1) and thus participates in the final round.

From Lemma A.2 it follows that candidate c is the unique winner in subelection (C, V_2) . So the final-stage election is $(\{c, w\}, V)$ and we have the following scores on the first two levels:

$$score_{(\{c, w\}, V)}^1(c) = score_{(\{c, w\}, V)}^1(w) = k + n,$$

$$score_{(\{c, w\}, V)}^2(c) = score_{(\{c, w\}, V)}^2(w) = 2k + 2n.$$

Since none of c and w have a strict majority on the first level, both candidates are level 2 BV winners in this two-candidate final-stage election. Hence, c has been prevented from being a unique BV winner by partition of voters in model TE.

From right to left: Assume that c can be prevented from being a unique BV winner by partition of voters in model TE. From Lemma A.2 we know that candidate c must participate in the final-stage election. Since we are in model TE, at most two candidates participate in the final run-off. To prevent c from being a unique BV winner of the final election, there must be another finalist and this other candidate has to beat or tie with c . Since w is the only candidate that can beat or tie with c in a two-candidate election, w has to move on to the final round to run against c . Let us say that c is the unique winner of subelection (C, V_2) and w is the unique winner of subelection (C, V_1) . For w to be the unique winner of subelection (C, V_1) , V_1 has to contain voters from the first voter group and w can win only on the $(n + 5)$ th level. In particular, x is placed before w in all voter groups except the first, so w can win in (C, V_1) only via voters from the first voter group participating in (C, V_1) . Moreover, since w is placed in the last or second-to-last position in all voters from the third and fourth groups, and since there is only one voter in the second group, w can win only on the $(n + 5)$ th level (which is w 's position in the votes from the first voter group).

Let $I \subseteq \{1, \dots, n\}$ be the set of indices i such that first-group voter

$$F_i (B - N[b_i]) H_i y w (N[b_i] \cup D \cup E \cup (F - F_i) \cup (H - H_i)) u v c x$$

belongs to V_1 . Let $\ell = \|I\|$. Since w is the unique level $n + 5$ BV winner of subelection (C, V_1) but y is placed before w in every vote in the first group, the one voter from the second group (which is the only voter who prefers w to y) must belong to V_1 . Thus we know that

$$score_{(C, V_1)}^{n+5}(w) = \ell + 1 \quad \text{and} \quad score_{(C, V_1)}^{n+4}(y) = score_{(C, V_1)}^{n+5}(y) = \ell.$$

For the candidates in B , we have

$$score_{(C, V_1)}^{n+4}(b_j) = score_{(C, V_1)}^{n+5}(b_j) = 1 + \|\{b_i \mid i \in I \text{ and } b_j \notin N[b_i]\}\|,$$

since each b_j scores one point up to the $(n + 4)$ th level from the voter in the second group and one point from the first group for every b_i with $i \in I$ such that $b_j \notin N[b_i]$ in graph G . Again, since w is the unique level $n + 5$ BV winner of subelection (C, V_1) , no $b_j \in B$ can score a point in each of the ℓ votes from the first voter group that belong to V_2 . This implies that for each $b_j \in B$ there has to be at least one b_i with $i \in I$ that is adjacent to b_j in G . Thus, the set B' of candidates b_i with $i \in I$ corresponds to a dominating set in G .

Recall that $\text{score}_{(C, V_1)}^{n+5}(w) = \ell + 1$ and $\text{score}_{(C, V_1)}^{n+4}(y) = \ell$. Note also that $\text{score}_{(C, V_1)}^{n+4}(b_j) \leq \ell$ for $1 \leq j \leq n$. Since w needs a strict majority to be a BV winner in subelection (C, V_1) , it must hold that $\text{maj}(V_1) \leq \ell + 1$. Since y and the $b_j \in B$ have a score of ℓ already one level earlier than w , it must hold that $\text{maj}(V_1) = \ell + 1$, which implies $\|V_2\| = 2\ell$ or $\|V_2\| = 2\ell + 1$. To ensure this cardinality of V_1 , other votes have to be added. Since y must not gain additional points from these votes up to the $(n + 5)$ th level, they cannot come from the fourth voter group. The remaining votes from the third voter group total up to $k - 1$. Thus, since w is the unique BV winner in subelection (C, V_1) , it must hold that $\ell \leq k$. So $\|B'\| = \ell \leq k$ and this means that there exists a dominating set of size at most k .

Observe that reducing from the W[2]-complete parameterized problem k -DOMINATING SET with the same construction and the new parameter $k' = 2k$ (the size of the smaller partition), we can prove that destructive control by partition of voters in model TE in Bucklin voting is W[2]-hard.

For the co-winner case, a fifth voter group has to be added consisting of one voter of the form $H F D E B w c x$ and this voter has to be part of V_1 . This ensures that w is the unique level 1 winner in the final election, thus preventing c successfully from being a winner. \square Theorem 3.21

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